The Ultimate AP® Calculus AB Unit 2 Cheat Sheet

2.1 Average vs. Instantaneous Rate

Average Rate of Change

Avg rate on [a, b] = [f(b) − f(a)]/(b − a) = slope of **secant line** thru (a, f(a)) & (b, f(b)); algebra only, no limit.

Ex: f(x) = x² on [1, 3] → (9 − 1)/(3 − 1) = 4. Units: output per input, e.g., ft/sec, gal/min, °F/hr, $/item.

Difference quotient: [f(a + h) − f(a)]/h = avg rate on [a, a + h], h ≠ 0 → a secant slope; it becomes f′(a) as h → 0.

Instantaneous Rate of Change

Inst. rate at a = lim of avg rates = **tangent** slope = f′(a).

Secant → tangent: h → 0 ⇒ secant slope → tangent slope.

Graph view: a secant cuts the curve at 2 points; a tangent touches at 1 point & matches the curve's direction there.

Average rate	Instantaneous rate
Over an interval	At 1 point
Secant slope	Tangent slope
Algebra only	Needs a limit
Avg. velocity	Velocity v(t)

In Context

Position s(t): avg velocity = Δs/Δt; inst. velocity v(t) = s′(t); units, e.g., m/sec; acceleration a(t) = v′(t).

Interpret f′(3) = 2: "At t = 3 hr, water level ↑ at 2 cm/hr." Always give time, value, units & direction (↑/↓).

Sign: rate > 0 → increasing; < 0 → decreasing; = 0 → momentarily constant; bigger |rate| → faster change.

Velocity vs. speed: v has sign (direction); speed = |v| ≥ 0.

Ex: s(t) = t² m → avg velocity on [1, 3] = (9 − 1)/2 = 4 m/sec; v(2) = s′(2) = 4 m/sec, same as the avg here.

⚠ Avg rate of change ≠ avg of values: use Δoutput/Δinput, not [f(a) + f(b)]/2, which is an avg value.

✓ FRQ "approximate rate at t = 4" from a table → difference quotient of nearby values; show setup.

2.2 Definition of the Derivative

Limit Definitions

f′(a) = $\lim_{h\to 0}$ [f(a + h) − f(a)]/h, if limit exists: finite & equal from both sides (h → 0⁻, h → 0⁺); else f′(a) DNE.

Alt. form: f′(a) = $\lim_{x\to a}$ [f(x) − f(a)]/(x − a) = slope of secant from (a, f(a)) to (x, f(x)) as x → a (x ≠ a).

f′(x) = $\lim_{h\to 0}$ [f(x + h) − f(x)]/h → slope function; exists only where limit exists (corners, cusps, breaks fail).

Recognize the Limit

Pattern: $\lim_{h\to 0}$ [(2 + h)³ − 8]/h = f′(2), f = x³ → 12.

Pattern: $\lim_{x\to\pi}$ (cos x + 1)/(x − π) = f′(π), f = cos x → 0.

By algebra: [(x + h)² − x²]/h = 2x + h → 2x as h → 0.

Notation

Notation	Read as
f′(x)	f prime of x
dy/dx	Derivative of y w.r.t. x
d/dx [f(x)]	Derivative of f(x)
y′	y prime
$dy/dx\big\|_{x=a}$	Value at x = a

Tangent Line Approximation

Tangent line at a: y = f(a) + f′(a)(x − a).

Ex: f = √x at 4: f′(x) = 1/(2√x) → f′(4) = 1/4.

Local linearization: f(x) ≈ f(a) + f′(a)(x − a) near a. Ex: √4.1 ≈ 2 + (1/4)(0.1) = 2.025 (true ≈ 2.0248).

Units of f′: output/input, e.g., gal/min.

⚠ f′(a) = NUMBER (slope); f′(x) = FUNCTION. Differentiate 1st, then plug in a; d/dx [f(a)] = 0 b/c it's a constant.

✓ Spot the limit → name f & a → differentiate. Most limit-def MCQs take < 30 sec; no algebra needed.

2.3 Estimating Derivatives

From a Table

Estimate f′(a) w/ closest points that bracket a: f′(a) ≈ [f(b) − f(c)]/(b − c), c < a < b; closer points → better estimate.

t (min)	0	2	5	9
W(t) (°F)	55	57	62	70

Ex: W′(3.5) ≈ (62 − 57)/(5 − 2) = 5/3 °F/min.

Ex: W′(1) ≈ (57 − 55)/(2 − 0) = 1 °F/min; W′(7) ≈ (70 − 62)/(9 − 5) = 2 °F/min → water warming faster later.

Symmetric difference: f′(a) ≈ [f(a + h) − f(a − h)]/2h → usually the best table estimate; centered at a.

One-sided: [f(a + h) − f(a)]/h or [f(a) − f(a − h)]/h when data exists on only 1 side (e.g., at the table's 1st or last t).

Interpret: "At t = 3.5 min, water temp is ↑ at ≈ 1.667 °F per min." Needs the time, the units & the direction.

From a Graph of f

Tangent slope: draw tangent at a, rise/run ≈ f′(a).

Sketch f′ from f: f ↑ → f′ > 0; f ↓ → f′ < 0; peaks & valleys → f′ = 0 (x-intercepts of f′); inflection pts → f′ max/min.

Line pieces: segment of f → f′ constant (horizontal piece); corner → f′ jumps (open dots at the corner x).

Rank slopes: steeper f → larger |f′|; flat → 0. Compare tangents to order f′ values, e.g., f′(1) < f′(3) < 0 < f′(5).

Read f′'s graph: f′ > 0 → f ↑; f′ < 0 → f ↓; f′ changes + to − → f peaks; f′ = 0 → flat (horizontal) tangent on f.

With a Calculator

nDeriv / d/dx: f′(a) to 3 decimals on calc parts.

Local linearity: zoom in → differentiable f looks like a line near a; its slope = f′(a); corners never straighten out.

⚠ Use the 2 values that straddle a, not table endpoints.

✓ Show (62 − 57)/(5 − 2); a bare decimal earns no FRQ credit, even if correct; give 3 decimals or a fraction.

2.4 Differentiability & Continuity

The Key Theorem

Differentiable ⇒ continuous: f′(a) exists → f continuous at a.

Converse FALSE: continuous ⇏ differentiable (|x| at 0).

Ex: f(x) = |x − 2| is continuous at x = 2, but left slope −1 ≠ right slope 1 → corner, not differentiable at 2.

Contrapositive: not continuous at a ⇒ no f′(a); jump, hole or vertical asymptote → no derivative there.

Where f′ Fails to Exist

Feature	Example	Why
Corner	|x| at 0	Slopes differ
Cusp	$x^{2/3}$ at 0	Slopes → ±∞
Vertical tangent	$\sqrt[3]{x}$ at 0	Slope infinite
Discontinuity	Jump, hole	Not continuous

Piecewise Functions

Step 1, continuity: left lim = right lim = f(a). **Step 2, smooth**: left slope = right slope at the x where pieces join.

Ex: f = x² (x ≤ 1), kx + m (x > 1). Continuous: 1 = k + m. Slopes: 2(1) = k → k = 2, m = −1 → f′(1) = 2.

One-sided derivatives: $\lim_{h\to 0^-}$ & $\lim_{h\to 0^+}$ of diff. quotient must be equal & finite; |x| at 0: −1 ≠ 1.

Vertical tangent: $\sqrt[3]{x}$ has f′ = $1/(3x^{2/3})$ → ∞ at 0.

Reading Graphs

Scan graph for sharp points, vertical tangents, breaks, holes; smooth & unbroken → differentiable there.

Endpoints: no 2-sided limit at a closed endpoint → theorems need "continuous on [a, b], differentiable on (a, b)."

⚠ "Continuous, so differentiable" = classic wrong answer; corners & cusps are continuous but not differentiable.

✓ Justify "not differentiable": show 1-sided slopes differ (e.g., −1 vs. 1 for |x|), or f not continuous at x = a.

2.5–2.6 Power Rule & Basic Rules

Power Rule

Power rule: d/dx [x^n] = nx^{n-1}, any constant n (integer, fraction, negative); e.g., $(x^5)' = 5x^4$, $(x^{-1})' = -x^{-2}$.

Rewrite	Then	Derivative
√x = $x^{1/2}$	$(1/2)x^{-1/2}$	$1/(2\sqrt{x})$
$1/x^3 = x^{-3}$	$-3x^{-4}$	$-3/x^4$
$\sqrt[3]{x^2} = x^{2/3}$	$(2/3)x^{-1/3}$	$2/(3\sqrt[3]{x})$
$5/x = 5x^{-1}$	$-5x^{-2}$	$-5/x^2$

Rewrite 1st: roots → fraction exponents; $1/x^n \to x^{-n}$.

Ex: y = 6√x − 2/x → $3x^{-1/2} + 2x^{-2}$ = 3/√x + 2/x².

Ex: y = $1/x^2 = x^{-2}$ → y′ = $-2x^{-3} = -2/x^3$; at x = 1, slope −2 & tangent line y = 1 − 2(x − 1), i.e., y = 3 − 2x.

Constant, Sum & Multiple

Rules: (c)′ = 0; (cf)′ = cf′; (f ± g)′ = f′ ± g′.

Ex: y = 4x³ − 2x + 7 → y′ = 12x² − 2; y′(1) = 10.

Simplify 1st: (x² + 3x)/x = x + 3 → y′ = 1, x ≠ 0.

Higher-order: f″ = (f′)′: x⁴ → 4x³ → 12x².

Tangent & Normal Lines

Normal line at a: ⊥ tangent, slope −1/f′(a).

Horizontal tangent: set f′(x) = 0 & solve. f = x³ − 3x → 3x² − 3 = 0 → x = ±1 → points (1, −2) & (−1, 2) on f.

Given slope: y = x², slope 6 → 2x = 6 → (3, 9).

Ex: tangent to y = x³ at x = 2: point (2, 8), slope 3(2)² = 12 → y = 8 + 12(x − 2), so (2.1)³ ≈ 9.2 (true 9.261).

⚠ Power rule needs variable base & constant exponent: d/dx [2^x] ≠ $x \cdot 2^{x-1}$; it's 2^x ln 2 (exponential rule).

✓ FRQ: leave answers unsimplified unless asked; simplifying risks losing a correct point.

2.7 Derivatives of sin, cos, e^x & ln x

Core Derivatives

f(x)	f′(x)	Note
sin x	cos x	Radians only
cos x	−sin x	Watch the minus
e^x	e^x	Its own derivative
ln x	1/x	x > 0
a^x	a^x ln a	Unit 3 extends
$\log_a x$	1/(x ln a)	Unit 3 extends

a^x: d/dx [10^x] = 10^x ln 10 (extra factor ln a).

Radians only: set calc to radian mode.

Why They Work

Special limits: $\lim_{h\to 0}$ (sin h)/h = 1 & $\lim_{h\to 0}$ (cos h − 1)/h = 0 → used to derive sin′ & cos′ via the limit def.

e ≈ 2.718: slope of e^x at 0 is 1 → $(e^x)' = e^x$.

Graph check: sin x has slope 1 at 0 (cos 0 = 1) & slope 0 at peaks, where cos x = 0; cos graph = slope graph of sin.

Examples

Mix: y = 3 sin x − $2e^x$ + 5 ln x → y′ = 3 cos x − $2e^x$ + 5/x.

Ex: y = 4 cos x − x → y′ = −4 sin x − 1.

Log rules 1st: ln(x^3) = 3 ln x → 3/x. ln(5x) = ln 5 + ln x → 1/x (ln 5 = constant); ln(1/x) = −ln x → −1/x.

Slopes: sin at π/3 → 1/2; ln at 2 → 1/2; e^x at ln 3 → 3.

Tangent to y = e^x at 0: slope e^0 = 1, point (0, 1) → y = x + 1.

⚠ (cos x)′ = −sin x; dropping the minus = most common error in this unit; check the sign on every cos term.

✓ $\lim_{h\to 0}$ [sin(π/6 + h) − sin(π/6)]/h = cos(π/6) = √3/2: recognize, don't compute; f = sin x, a = π/6.

2.8–2.9 Product & Quotient Rules

Product Rule

Product: (fg)′ = f′g + fg′ → "1st′ · 2nd + 1st · 2nd′." Ex: (x^2 sin x)′ = 2x sin x + x^2 cos x; × (2x)(cos x).

Ex: (xe^x)′ = e^x + xe^x = e^x(1 + x), so = 0 at x = −1.

Ex: y = x^3e^x → y′ = x^2e^x(x + 3) → y′ = 0 at x = 0, −3 ($e^x \neq 0$).

3 factors: (fgh)′ = f′gh + fg′h + fgh′ (1 prime per term).

Quotient Rule

Quotient: (f/g)′ = (f′g − fg′)/g^2, g ≠ 0 → "low d-high − high d-low, over low²"; leave g^2 factored, don't expand.

Ex: (sin x/x)′ = (x cos x − sin x)/x^2, x ≠ 0.

Ex: y = e^x/x → (xe^x − e^x)/x^2 = e^x(x − 1)/x^2.

Order matters: numerator subtracts → f′g − fg′ ≠ fg′ − f′g; start w/ derivative of top; sign flip = common wrong choice.

Tables of Values

x	f	f′	g	g′
2	3	−1	4	5
5	−2	3	1	−4

h = fg: h′(2) = (−1)(4) + (3)(5) = 11. **k = f/g**: k′(2) = [(−1)(4) − (3)(5)]/4^2 = −19/16; never g′ in the denominator.

p = x·f(x): p′(5) = f(5) + 5f′(5) = −2 + 15 = 13.

q = g/x: q′(5) = [5g′(5) − g(5)]/5^2 = −21/25.

Tangent to fg at 5: value (−2)(1) = −2, slope (3)(1) + (−2)(−4) = 11 → y + 2 = 11(x − 5) thru point (5, −2).

From graphs: read f(a), g(a) off the curves; f′(a), g′(a) = slopes of line pieces at a; at a corner, f′(a) DNE.

Skip the rule: $x^3/x = x^2$; (2x + 1)/x = 2 + 1/x; constant denom (x² + 1)/5 → 2x/5; 3/x = $3x^{-1}$ → $-3/x^2$.

⚠ (fg)′ ≠ f′g′ & (f/g)′ ≠ f′/g′; derivatives split over sums only.

✓ Write the rule w/ letters 1st, then substitute table values; setup earns partial credit even w/ arithmetic slips.

2.10 Other Trig Derivatives

Tan, Cot, Sec & Csc

f(x)	f′(x)
tan x	$\sec^2 x$
cot x	$-\csc^2 x$
sec x	sec x tan x
csc x	−csc x cot x

Co- rule: cos, cot, csc → NEGATIVE derivatives; tan, sec (non-co) → positive; sin → +cos, cos → −sin.

Pairs: tan/$\sec^2$, cot/$-\csc^2$, sec/sec tan, csc/−csc cot.

Derive w/ Quotient Rule

tan x = sin x/cos x → (cos²x + sin²x)/$\cos^2 x$ = 1/$\cos^2 x$ = $\sec^2 x$ (sin² + cos² = 1); FRQs may ask you to show it.

sec x = 1/cos x → (0·cos x + sin x)/$\cos^2 x$ = (1/cos x)(sin x/cos x) = sec x tan x after splitting the fraction.

Domains: tan & sec undefined where cos x = 0 (x = π/2 + kπ) → no derivative there; same for cot & csc at x = kπ.

Unit Circle Values

x	sin	cos	tan
0	0	1	0
π/6	1/2	√3/2	1/√3
π/4	√2/2	√2/2	1
π/3	√3/2	1/2	√3
π/2	1	0	Undef.

Ex: tan at π/4: slope $\sec^2(\pi/4) = 2$ → y − 1 = 2(x − π/4).

Ex: y = sec x at π/3 → sec(π/3)·tan(π/3) = 2·√3 = 2√3.

Ex: (5 sec x − cot x)′ = 5 sec x tan x + $\csc^2 x$.

Mix: (x tan x)′ = tan x + x $\sec^2 x$ (product rule).

⚠ $\sec^2 x$ = (sec x)², not sec(x^2); square the output.

✓ Memorize all 6 trig derivatives & unit circle values; the AP Calc exam gives no formula sheet at all.