The Ultimate AP® Calculus AB Unit 3 Cheat Sheet

3.1 The Chain Rule

The Rule

Chain rule: $d/dx[f(g(x))] = f'(g(x))\cdot g'(x)$: derivative of the outside (inside kept) × derivative of the inside.
Leibniz form: $y = f(u)$, $u = g(x) \rightarrow dy/dx = (dy/du)\cdot(du/dx)$; the du's "cancel." Ex: $y = u^3$, $u = 2x + 1 \rightarrow 3u^2\cdot 2$.

Spot it: a function inside another: $(3x + 1)^5$, $\sin(x^2)$, e^{2x}, $\ln(x^2 + 1)$, $\sqrt{(4 - x)}$; anything that isn't plain x inside.
Composite check: $(f \circ g)(x) = f(g(x))$; the chain rule applies whenever the inside isn't just x, even for $2x$ or $x + 1$.

Constants: e^5, ln 3, $\sin(\pi/6)$ are numbers → derivative 0.

Common Chain Forms

Function	Derivative
$[g(x)]^n$	$n[g(x)]^{n-1}\cdot g'(x)$
$\sqrt{g(x)}$	$g'(x)/(2\sqrt{g(x)})$
$e^{g(x)}$	$e^{g(x)}\cdot g'(x)$
ln g(x)	$g'(x)/g(x)$
sin g(x)	$\cos g(x)\cdot g'(x)$
cos g(x)	$-\sin g(x)\cdot g'(x)$
tan g(x)	$\sec^2 g(x)\cdot g'(x)$
$a^{g(x)}$	$a^{g(x)}\cdot \ln a\cdot g'(x)$

Worked Examples

Power: $(3x + 1)^5 \rightarrow 5(3x + 1)^4\cdot 3 = 15(3x + 1)^4$.
Trig: $\sin(x^2) \rightarrow \cos(x^2)\cdot 2x$. **Exp**: $e^{-4x} \rightarrow -4e^{-4x}$. **Log**: $\ln(x^2 + 1) \rightarrow 2x/(x^2 + 1)$. **Tan**: $\tan(5x) \rightarrow 5\sec^2(5x)$.
Root: $\sqrt{(4 - x)} = (4 - x)^{1/2} \rightarrow \frac{1}{2}(4 - x)^{-1/2}\cdot(-1) = -1/(2\sqrt{(4 - x)})$.
Power of trig: $\cos^4 x = (\cos x)^4 \rightarrow 4\cos^3 x\cdot(-\sin x) = -4\cos^3 x \sin x$; the power is the outermost layer.

⚠ Keep the inside unchanged in f': d/dx sin(3x) = 3cos(3x), not cos(3)·3 or cos(x)·3; the inside rides along.

✓ Work outside → in: peel 1 layer at a time & multiply by each inner derivative until you reach plain x.

3.1 Chain Rule: Tables, Graphs & Layers

From a Table

Setup: $h(x) = f(g(x)) \rightarrow h'(a) = f'(g(a))\cdot g'(a)$; find g(a) first, then look up f' at THAT value, not at a.

x	f(x)	f'(x)	g(x)	g'(x)
1	3	−2	2	4
2	5	6	1	−3
3	4	1	3	2

Ex: $h(x) = f(g(x)) \rightarrow h'(1) = f'(g(1))\cdot g'(1) = f'(2)\cdot 4 = 6\cdot 4 = 24$.
Ex: $k(x) = g(f(x)) \rightarrow k'(2) = g'(f(2))\cdot f'(2) = g'(5)\cdot 6 \rightarrow$ need g'(5); read every row & column of the table carefully.
Self-composite: d/dx f(f(x)) = f'(f(x))·f'(x) → at x = 1: f'(f(1))·f'(1) = f'(3)·(−2) = 1·(−2) = −2 from the table.
Power of g: d/dx [g(x)]³ at x = 2 → $3[g(2)]^2\cdot g'(2) = 3(1)^2(-3) = -9$; the table gives both g & g' at x = 2.

From a Graph

Slopes: f'(a) = slope of the graph of f at x = a; on a line segment, slope = rise/run of that whole segment.
Corner: if g(a) lands on a corner or cusp of f, f'(g(a)) doesn't exist → h'(a) DNE (unless g'(a) = 0 cases).
Graph + table mix: read f' from the graph's slope & g, g' from the table; label which source gives each number.

Multiple Layers & Mixed Rules

3 layers: $\sin^3(2x) = [\sin(2x)]^3 \rightarrow 3\sin^2(2x)\cdot\cos(2x)\cdot 2 = 6\sin^2(2x)\cos(2x)$; 1 factor per layer, 3 in all.
Product + chain: $x^2e^{3x} \rightarrow 2xe^{3x} + 3x^2e^{3x} = xe^{3x}(2 + 3x)$.
Quotient + chain: $\sin(2x)/x \rightarrow [2x\cos(2x) - \sin(2x)]/x^2$.
FRQ write-up: h'(1) = f'(g(1))·g'(1) = f'(2)·4 = 6·4 = 24: one clear chain of equalities earns the point.

⚠ f'(g(a)) ≠ f'(a)·g(a): evaluate g first, then f' at g(a), then multiply by g'(a); order matters every time.

✓ Write h'(a) = f'(g(a))·g'(a) before plugging in; label every number you read from the table or the graph.

3.2 Implicit Differentiation

When & Why

Implicit: x & y mixed ($x^2 + y^2 = 25$, $xy + y^3 = 5$) → differentiate both sides w/o solving for y first; often impossible.
Key: y is a function of x, so every y term gets a dy/dx by the chain rule: $d/dx(y^3) = 3y^2\cdot dy/dx$; x terms don't.

Steps

1. d/dx both sides. **2.** Tack dy/dx onto each y-derivative. **3.** Move dy/dx terms to 1 side, factor, divide.

Term	d/dx
y^2	$2y\cdot y'$
xy	$y + x\cdot y'$
sin y	$\cos y\cdot y'$
e^y	$e^y\cdot y'$
ln y	y'/y

Worked Examples

Circle: $x^2 + y^2 = 25 \rightarrow 2x + 2y\cdot y' = 0 \rightarrow y' = -x/y$; at (3, 4): slope = −3/4; at (3, −4): slope = +3/4.
Product: $xy + y^3 = 10 \rightarrow y + xy' + 3y^2y' = 0 \rightarrow y'(x + 3y^2) = -y \rightarrow y' = -y/(x + 3y^2)$; at (9, 1): y' = −1/12.
Trig implicit: $\sin y + x = y^2 \rightarrow \cos y\cdot y' + 1 = 2y\cdot y' \rightarrow y' = 1/(2y - \cos y)$; collect all the y' terms on 1 side first.
Tangent line at (3, 4) on the circle: $y - 4 = -\frac{3}{4}(x - 3)$.
Normal line slope = −1/(dy/dx): at (3, 4) on the circle → 4/3.

Horizontal & Vertical Tangents

Horizontal: numerator of dy/dx = 0 (& denominator ≠ 0).
Vertical: denominator = 0 (& numerator ≠ 0).
Circle: $y' = -x/y \rightarrow$ horizontal tangents at x = 0 → (0, ±5); vertical tangents at y = 0 → (±5, 0); check both.

⚠ d/dx(xy) needs the PRODUCT rule: y + x·y', not just y'.

✓ dy/dx may contain both x & y: plug in BOTH coordinates of the point, never just the x-value alone.

3.2 Implicit: Second Derivatives & FRQ Moves

Finding d^2y/dx^2

Method: differentiate dy/dx again (quotient rule), then substitute dy/dx back in & simplify w/ the original curve.
Circle: $y' = -x/y \rightarrow y'' = -(y - x\cdot y')/y^2 = -(y + x^2/y)/y^2 = -(x^2 + y^2)/y^3 = -25/y^3$ (since $x^2 + y^2 = 25$).
Use the original equation to simplify: $x^2 + y^2 = 25$ turned the whole numerator into 25; look for that move.
At (3, 4): y'' = −25/64 < 0 → the curve is concave down there; at (3, −4): +25/64 → concave up.
FRQ ex: $x^2 + xy + y^2 = 7$ at (1, 2): $2x + y + xy' + 2yy' = 0 \rightarrow 4 + 5y' = 0 \rightarrow y' = -4/5$ at that point (1, 2).

Typical FRQ Parts

Part asks	Move
Show dy/dx = ...	Implicit diff + algebra
Tangent line at a pt	Plug both coords → slope
Horizontal tangent	Set numerator = 0 + check
Vertical tangent	Set denominator = 0
Concavity at a pt	Find y'', plug in

Horizontal tangent check: set the numerator = 0, then plug back into the original curve to find real points.
"Show that" parts: write every step; the given answer earns nothing unless your own work clearly reaches it.

Related Ideas

Linear approximation later uses this slope: $y \approx y_1 + y'(x - x_1)$ near the point (Unit 4 & many FRQs ask it).
dx/dy = 1/(dy/dx): useful when the curve is easier to write as x in terms of y; vertical tangent ↔ dx/dy = 0.
Concavity from y'': the sign of y'' at the point → concave up (+) or concave down (−) there; also for FRQ justifications.

⚠ Don't drop the y' inside y'': every y term in y' still needs the chain rule again when you differentiate it.

✓ Substitute the given point as early as allowed to save algebra (but find y'' symbolically first).

3.3 Derivatives of Inverse Functions

The Formula

Inverse derivative: $(f^{-1})'(a) = 1 / f'(f^{-1}(a))$: slope of the inverse = reciprocal slope at the matching point.
Why: $f(f^{-1}(x)) = x \rightarrow$ chain rule → $f'(f^{-1}(x))\cdot(f^{-1})'(x) = 1$.
Reflection: if (b, a) is on f, then (a, b) is on f^{-1} & the tangent slopes there are reciprocals (m ↔ 1/m).

Steps

1. Find $b = f^{-1}(a)$: solve f(b) = a (often a table or a guess). **2.** Compute f'(b). **3.** Take 1/f'(b) as the answer.

x	f(x)	f'(x)
1	2	4
2	7	9
3	15	10
4	20	12

Table ex: $(f^{-1})'(7) \rightarrow f(2) = 7 \rightarrow f'(2) = 9 \rightarrow (f^{-1})'(7) = 1/9$.
Formula ex: $f(x) = x^3 + x$, find $(f^{-1})'(2)$: $f(1) = 2 \rightarrow f'(x) = 3x^2 + 1 \rightarrow f'(1) = 4 \rightarrow (f^{-1})'(2) = 1/4$ exactly (no algebra).
Exp ex: $f(x) = e^x + 2x \rightarrow f(0) = 1 \rightarrow (f^{-1})'(1) = 1/f'(0) = 1/(e^0 + 2) = 1/3$; you never need a formula for f^{-1}.
Graph method: $(f^{-1})'(a) = 1/$(slope of f where y = a); read that slope off f's tangent line or its graph directly.

Classic Cases

ln x is the inverse of e^x: $d/dx \ln x = 1/e^{\ln x} = 1/x$; the same idea gives √x & every inverse trig rule.
Existence: needs f one-to-one near the point & f'(b) ≠ 0 (else f^{-1} has a vertical tangent there, no derivative).

⚠ $(f^{-1})'(a) \neq 1/f'(a)$: evaluate f' at $f^{-1}(a)$, the x-value that gives a, not at a itself; it's the #1 trap here.

✓ Always find which x makes f(x) = a first; it's usually a small integer, a table entry or a quick guess.

3.4 Inverse Trig Derivatives

Memorize These

Function	Derivative	Domain
arcsin x	$1/\sqrt{(1 - x^2)}$	$\lvert x\rvert < 1$
arccos x	$-1/\sqrt{(1 - x^2)}$	$\lvert x\rvert < 1$
arctan x	$1/(1 + x^2)$	all real x
arccot x	$-1/(1 + x^2)$	all real x
arcsec x	$1/(\lvert x\rvert\sqrt{(x^2 - 1)})$	$\lvert x\rvert > 1$

Notation: arcsin x = $\sin^{-1}$ x (inverse), NOT 1/sin x = csc x.
Co-function pattern: arccos, arccot & arccsc derivatives are the negatives of arcsin, arctan & arcsec.
Most tested: arcsin & arctan w/ the chain rule; know arccos as just the negative of arcsin's derivative.

With the Chain Rule

Function	Derivative
arcsin u	$u'/\sqrt{(1 - u^2)}$
arctan u	$u'/(1 + u^2)$

Ex: $\arctan(3x) \rightarrow 3/(1 + 9x^2)$. **Ex**: $\arcsin(x^2) \rightarrow 2x/\sqrt{(1 - x^4)}$.
Ex: x·arctan x → arctan x + $x/(1 + x^2)$ (product rule first).
Scaled input: $\arcsin(x/3) \rightarrow (1/3)/\sqrt{(1 - x^2/9)} = 1/\sqrt{(9 - x^2)}$.
Tangent line: y = arctan x at x = 1 → point (1, π/4), slope $1/(1 + 1^2) = \frac{1}{2}$ → tangent: $y - \pi/4 = \frac{1}{2}(x - 1)$.

Values to Know

Outputs: arcsin ½ = π/6; arccos ½ = π/3; arctan 1 = π/4; arctan √3 = π/3; arcsin 1 = π/2; arccos 0 = π/2.
Ranges: arcsin → [−π/2, π/2]; arctan → (−π/2, π/2); arccos → [0, π]; pick the angle in that range, not any angle.
Derivation: y = arcsin x → sin y = x → cos y·y' = 1 → y' = 1/cos y = $1/\sqrt{(1 - x^2)}$ (implicit diff + identity).

⚠ Square the WHOLE inner function: arctan(2x) → $2/(1 + 4x^2)$, not $2/(1 + 2x^2)$; then × the inner derivative.

✓ Derivatives of inverse trig are algebraic (no trig left): a fast check to eliminate answer choices.

3.5 Selecting the Right Procedure

Pick the Rule

Expression looks like	Use
Sum/difference	Term by term
f·g	Product rule
f/g	Quotient (or rewrite)
Function inside function	Chain rule
x & y mixed	Implicit diff
f^{-1} at a point	$1/f'(f^{-1}(a))$
Messy products/powers	Log properties first

Simplify First

Rewrite: $5/x^3 = 5x^{-3} \rightarrow -15x^{-4}$; skip the quotient rule when the bottom is a constant or a monomial.
Log rules: $\ln[(x^2)(x + 1)^3] = 2\ln x + 3\ln(x + 1) \rightarrow 2/x + 3/(x + 1)$; expand logs before differentiating.
Logarithmic diff.: $y = x^x \rightarrow \ln y = x \ln x \rightarrow y'/y = \ln x + 1 \rightarrow y' = x^x(\ln x + 1)$; variable base & exponent.

Order of Rules

Outermost operation first: $(x \sin x)^3 \rightarrow$ chain first (cube), then product inside: $3(x \sin x)^2(\sin x + x \cos x)$.
Quotient → product: f/g = $f\cdot g^{-1}$ works too: product + chain gives the same answer, often w/ less algebra.

Common Traps

$d/dx(e^2)$ = 0 (a constant). **$d/dx(x^e)$** = ex^{e-1}. **$d/dx(e^x)$** = e^x.
Other bases: $\log_b x \rightarrow 1/(x \ln b)$; $b^x \rightarrow b^x \ln b$; so $2^x \rightarrow 2^x \ln 2$ & $\log_{10} x \rightarrow 1/(x \ln 10)$; ln x is just the b = e case.

⚠ Product rule ≠ product of derivatives: (fg)' = f'g + fg', never f'g'; quotient: (f'g − fg')/g², bottom squared.

✓ Name the outermost operation out loud; it tells you the first rule to apply, then work inward layer by layer.

3.6 Higher-Order Derivatives & Exam Moves

Notation

2nd derivative: f''(x) = y'' = d^2y/dx^2; **nth**: $f^{(n)}(x) = d^ny/dx^n$; just differentiate f' again, then again, as needed.
Ex: $f(x) = x^4 - 3x^2 \rightarrow f' = 4x^3 - 6x \rightarrow f'' = 12x^2 - 6 \rightarrow f''' = 24x \rightarrow f^{(4)} = 24 \rightarrow f^{(5)} = 0$ (polynomials end).

Patterns

f(x)	Pattern
sin x	sin → cos → −sin → −cos (repeats every 4)
e^{kx}	$f^{(n)} = k^ne^{kx}$
ln x	1/x, $-1/x^2$, $2/x^3$, $-6/x^4$ ...
x^n (n whole #)	0 after n + 1 derivatives

Ex: $d^{27}/dx^{27}(\sin x)$: 27 = 4·6 + 3 → same as the 3rd derivative of sin x = −cos x; use the remainder mod 4.
Product pattern: $(xe^x)^{(n)} = (x + n)e^x$ for every n.

Meaning

f'' = rate of change of f': f'' > 0 → f' increasing → concave up; f'' < 0 → concave down (Unit 5 details).
Motion preview: position s → velocity s' → acceleration s'' (used heavily in Unit 4 particle-motion Qs).
Implicit y'' (from 3.2) is a classic higher-order Q: find y', differentiate again, substitute y' back in.

Exam Format

MCQ: 42 Qs: 29 w/o calculator (62 min) + 13 w/ graphing calculator (38 min); Unit 3 ≈ 9–13% of AB MCQs.
FRQ: 6 Qs (2 calc, 4 no-calc); chain, implicit & inverse derivatives show up inside many of them.

⚠ f'' is the derivative of f', not (f')²; don't square the first derivative or multiply f' by itself to get it.

✓ No-calc section: show the chain-rule factor clearly; unsimplified correct answers still earn credit.