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# The Ultimate AP® Precalculus Unit 1 Cheat Sheet

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## 1.1–1.3 Change in Tandem & Rates of Change

### 1.1 Change in Tandem

**Function**: each input → exactly 1 output. Input & output values vary together (in tandem) as the input changes.
**Increasing**: x↑ → f(x)↑. **Decreasing**: x↑ → f(x)↓. State the intervals using x-values, never the y-values.
**Concave up**: rate of change ↑ (graph bends up). **Concave down**: rate of change ↓ (graph bends down).
**Zeros**: inputs where f(x)=0 → x-intercepts of the graph.
**Rising vs. rising faster**: f can ↑ while its rate ↓ (concave down: still climbing, just more slowly) on every interval.

### 1.2 Average Rate of Change

**AROC** on [a, b] = [f(b) − f(a)] / (b − a) = slope of the **secant line** thru the points (a, f(a)) & (b, f(b)) on the graph.
**Rate at a point**: approx. w/ AROC over shorter & shorter intervals around it (secant → tangent slope).
**Sign**: rate > 0 → f increasing; rate < 0 → f decreasing.
**Units**: output units per input unit (m/s, $/yr). Always include the units when you interpret in context of any AROC answer.
Estimate the rate at x = 2: [f(2.1) − f(1.9)] / 0.2, using a small interval centered on x = 2 (symmetric).

### 1.3 Linear vs. Quadratic

| Function | Over equal Δx | Graph |
|---|---|---|
| Linear | 1st diffs const | No concavity |
| Quadratic | 2nd diffs const | Concave 1 way |
| Cubic | 3rd diffs const | Can switch |

**Example**: f(x) = x² at x = 0, 1, 2, 3 → 0, 1, 4, 9 → 1st diffs 1, 3, 5 → 2nd diffs 2, 2 → a quadratic by its 2nd differences.
**Quadratic AROC** changes at a const rate → AROCs over consecutive equal intervals ↑ → concave up.
**Linear**: f(x) = mx + b → AROC = m on every interval; its graph is a line w/ no concavity at all, whatever interval you pick.

⚠ AROC = Δoutput ÷ Δinput. Never average the two outputs; that's a mean value, not a rate.

✓ Table Qs: check that Δx is equal before using differences.

## 1.4 Polynomial Functions & Rates of Change

### Form & Vocabulary

**Polynomial**: $p(x) = a_nx^n + a_{n-1}x^{n-1} + \ldots + a_1x + a_0$, w/ whole-number powers (0, 1, 2…) only; no x in a denominator.
**Degree** n = highest power of x. **Leading coeff.** $a_n \neq 0$.
Constant term $a_0$ = the y-intercept, p(0), always.

### Extrema

**Local max**: f switches inc → dec. **Local min**: dec → inc. The rate of change switches sign at each one.
**Global (absolute) max/min**: the greatest/least output value over the function's entire domain, not just nearby.
**Even degree** → has a global max OR a global min (never both). **Odd degree** → has neither (ends opposite).
**Degree n** → at most n − 1 local extrema & at most n − 2 inflection points (fewer is possible too).
**Turning pts**: cubic → 0 or 2; quartic → 1 or 3; the exact count depends on the polynomial's coeffs (it varies case by case).

### Inflection & Concavity

**Point of inflection**: the rate of change switches inc ↔ dec → concavity flips (up ↔ down) there on the graph.

| Feature | Graph | Rate of change |
|---|---|---|
| Increasing | Rises L → R | Positive |
| Local max | Peak | + → − |
| Local min | Valley | − → + |
| Concave up | Bends up | Increasing |
| Inflection pt | Bend flips | Max/min rate |

**Differences**: a degree n polynomial → $n^{th}$ differences are constant over equal-length input intervals.

⚠ Decreasing + concave up ≠ increasing. Concavity is about the rate of change, not the outputs.

✓ Justify extrema in words: "f changes from increasing to decreasing at x = 2, so f has a local max."

## 1.5 Polynomial Functions & Complex Zeros

### Zeros, Factors, Intercepts

**Real zero** a ⟺ (x − a) is a factor ⟺ (a, 0) is an x-intercept.
**Fundamental Thm of Algebra**: degree n → exactly n complex zeros (counting each by its multiplicity).
**Non-real zeros** come in conjugate pairs: a + bi → a − bi is a zero too (when all coefficients are real).
**Graph shows** real zeros only; non-real zeros never appear as x-intercepts, so count them separately on any graph.
**Quadratic formula**: x = [−b ± √(b² − 4ac)]/(2a); if b² − 4ac < 0 → 2 non-real (conjugate) zeros from the formula.
**Example**: x³ − x² + 9x − 9 = x²(x − 1) + 9(x − 1) = (x − 1)(x² + 9) → zeros 1 & ±3i (by grouping) from the factored form.

### Multiplicity

| Multiplicity | At x-axis | Sign of f |
|---|---|---|
| 1 (odd) | Crosses | Switches |
| 2, 4 (even) | Touches, bounces | Same |
| 3, 5 (odd ≥3) | Crosses, flattens | Switches |

**Build from zeros**: 2 (mult 2), −1, 3i → p(x) = a(x − 2)²(x + 1)(x² + 9). Use a given point to solve for a w/ the given data.

### Inequalities & Symmetry

**Sign chart**: between consecutive real zeros, p stays all + or all −. Test 1 x-value in each interval & record its sign.
**Even fn**: f(−x) = f(x) → symmetric over the y-axis. **Odd fn**: f(−x) = −f(x) → symmetric about the origin (180° rotation).
**Check**: x⁴ − 3x² is even; x³ + x is odd; x³ + 1 is neither (the constant term breaks the odd symmetry).
**Example**: (x + 1)(x − 3) < 0 → test −2 (+), 0 (−), 4 (+) → solution: −1 < x < 3 (zeros excluded) from the sign chart.

⚠ Degree 5 w/ 3 real zeros → the other 2 zeros are non-real conjugates, not "missing" zeros of the polynomial.

✓ p(x) ≥ 0: include the zeros themselves in your answer; for p(x) > 0, exclude them instead on the number line.

## 1.6 Polynomials & End Behavior

### Leading Term Rules

**End behavior**: as |x| gets large, the **leading term** $a_nx^n$ dominates p(x). **LC** = leading coefficient.

| Degree, LC | x → −∞ | x → ∞ |
|---|---|---|
| Even, LC > 0 | p → ∞ | p → ∞ |
| Even, LC < 0 | p → −∞ | p → −∞ |
| Odd, LC > 0 | p → −∞ | p → ∞ |
| Odd, LC < 0 | p → ∞ | p → −∞ |

### Limit Notation

**Read**: $\lim_{x\to\infty} p(x) = \infty$ means "as x increases w/o bound, p(x) increases w/o bound" (right end).
**Example**: p(x) = −2x³ + 5x² + 1 → odd, $a_n < 0$ → $\lim_{x\to-\infty} p(x) = \infty$ & $\lim_{x\to\infty} p(x) = -\infty$ (ends go opposite ways).

**Factored form**: multiply the leading terms. −x(x − 1)²(x + 4) → −x⁴ → even, LC < 0 → both ends fall ↓.
**Other end**: $\lim_{x\to-\infty} p(x) = -\infty$ means as x decreases w/o bound, p(x) decreases w/o bound on the graph's left end.

### Connecting Features

**Odd degree**: ends go opposite ways → at least 1 real zero; its range is all real numbers (by end behavior alone).
**Even degree**: ends match → may have 0 real zeros (e.g., x² + 1 never reaches the x-axis at all) on any graph.
**Sketch order**: end behavior → zeros & their multiplicity → y-intercept → connect w/ a smooth curve through every point.
**Even degree, $a_n > 0$** → a global min exists; $a_n < 0$ → a global max exists instead (it can't have both).

⚠ End behavior ignores lower terms only for large |x|. Near x = 0, the lower terms matter a lot of the graph.

✓ MCQ graph match: check the ends + the y-intercept first; that eliminates most choices fast on MCQs.

## 1.7–1.8 Rational Functions: End Behavior & Zeros

### Compare Degrees (1.7)

**Rational fn**: r(x) = p(x)/q(x), polynomials, q(x) ≠ 0. Its ends follow the ratio of the leading terms of p & q.

| Degrees | End behavior | Example |
|---|---|---|
| deg p < deg q | HA y = 0 | 3x/(x² + 1) |
| deg p = deg q | HA y = a/b | 2x²/(x² − 9)→2 |
| deg p = deg q + 1 | Slant asymptote | (x² + 1)/x |
| deg p > deg q + 1 | Like a poly | x⁴/(x − 1) |

**Slant asymptote**: long divide; the quotient line = the asymptote. (x² + 1)/x = x + 1/x → slant y = x for large |x|.
**Limit form**: HA y = 2 means $\lim_{x\to\pm\infty} r(x) = 2$. The graph CAN cross an HA for small |x| (HA = end behavior only).
**Why**: (2x² + 3)/(x² − 9) ≈ 2x²/x² = 2 for large |x| → HA y = 2.
**Rate near an HA**: the graph flattens out → its rate of change → 0 as x → ±∞, so the curve levels off.

### Zeros (1.8)

**Zeros of r**: real zeros of p that are in r's domain (not also zeros of q, which give holes instead) in the original form.
**Example**: r(x) = (x² − 4)/(x² − x − 6) = (x − 2)(x + 2)/[(x − 3)(x + 2)] → only zero is x = 2 (x = −2 is a hole).
**Domain**: all reals except the zeros of q. Always read it from the ORIGINAL form, before canceling.
**Sign at zeros**: odd mult. in p → r flips sign there; even mult. → r touches the axis, keeps its sign at that zero.

### Rational Inequalities

**Sign chart**: mark zeros of p AND of q; the sign can switch at each. Never include the zeros of q in your final answer.
**Example**: (x − 1)/(x + 2) ≥ 0 → x < −2 or x ≥ 1 (−2 excluded b/c q(−2) = 0; 1 included b/c r(1) = 0) in the solution set.

⚠ A zero of the numerator that also zeros the denominator = a hole, not an x-intercept on the graph of r.

✓ Factor numerator & denominator fully before naming any feature: zeros, VAs, holes or HAs on the graph.

## 1.9–1.10 Vertical Asymptotes & Holes

### Vertical Asymptotes (1.9)

**VA at x = a**: a is a zero of q w/ a higher multiplicity in q than in p (the factor doesn't fully cancel) in the expression.
**Near a VA**: $\lim_{x\to a^-} r(x)$ and $\lim_{x\to a^+} r(x)$ are each ∞ or −∞.
**Side behavior**: odd mult. in q (after canceling) → opposite signs on each side; even → same sign of the asymptote.
**Odd**: 1/(x − 1) → x→1⁻: −∞; x→1⁺: ∞. **Even**: 1/(x − 1)² → ∞ on both sides of x = 1 (same sign on each side).
**Sign test**: plug a value just left & right of a (e.g., 2.9 & 3.1 for a = 3) to get r's sign on each side of the vertical asymptote.

### Holes (1.10)

**Hole at x = a**: a is a zero of p & q w/ mult. in p ≥ mult. in q.
**Hole's y**: cancel the common factor, then plug a into the simplified r. $\lim_{x\to a} r(x)$ = that value (finite).

| Zero a of q | Feature | Limit at a |
|---|---|---|
| Not a zero of p | VA | ±∞ |
| Mult. q > mult. p | VA | ±∞ |
| Mult. p ≥ mult. q | Hole | Finite |

**Both at once?** (x − 1)/(x − 1)² = 1/(x − 1) → mult. in q is bigger → VA at x = 1, no hole there for this rational function.
**Calculator**: holes don't show on the screen. Find them algebraically from the factored form of r.

### Worked Example

**r(x)** = (x² − 4)/(x² − x − 6) → simplified (x − 2)/(x − 3), x ≠ −2.
**Features**: hole (−2, 4/5); VA x = 3; zero x = 2; HA y = 1 (equal degrees); y-intercept (0, 2/3) from the simplified form.
**Near x = 3**: x → 3⁺ → r → ∞; x → 3⁻ → r → −∞ (multiplicity 1, so the sign flips) on opposite sides of x = 3.
**Domain**: x ≠ −2, 3 → every zero of q is excluded, VA or hole.

⚠ Canceling does not fix the domain: x = −2 is still excluded from r, even after you simplify it.

⚠ A graph never crosses a VA, but it can cross an HA.

## 1.11–1.12 Equivalent Forms & Transformations

### Pick the Right Form (1.11)

**Standard form** → degree, end behavior, y-intercept.
**Factored form** → zeros, VAs, holes, sign changes.
**Long division**: p(x) = d(x)·q(x) + r(x), w/ deg r < deg d. It reveals the slant asymptote of a rational fn.
**Binomial Thm**: $(a + b)^n$ coeffs = row n of **Pascal's triangle** (1 3 3 1 for n = 3; 1 4 6 4 1 for n = 4) (binomial coefficients).
**Example**: (x + 2)³ = x³ + 3(x²)(2) + 3(x)(4) + 8 = x³ + 6x² + 12x + 8, using row 3: 1 3 3 1 of Pascal's triangle.
**Divide**: (2x² + 3x + 5)/(x + 1) = 2x + 1 + 4/(x + 1) → slant asymptote y = 2x + 1 (the quotient) for large values of |x|.
**Pascal row 4**: 1 4 6 4 1 → (x − 1)⁴ = x⁴ − 4x³ + 6x² − 4x + 1.

### g(x) = a·f(b(x − h)) + k (1.12)

| Parameter | Effect on graph |
|---|---|
| a (outside) | Vert. dilation by \|a\| |
| a < 0 | Reflect over x-axis |
| k (outside) | Shift up k |
| b (inside) | Horiz. dilation 1/\|b\| |
| b < 0 | Reflect over y-axis |
| h (inside) | Shift right h |

**Inside vs. outside**: inside changes act on x (and feel reversed); outside changes act on y directly.
**Domain/range**: horizontal shifts & dilations alter the domain; vertical ones alter the range of the new function g.
**Point mapping**: (x, y) on f → (x/b + h, a·y + k) on g.
**Example**: g(x) = −2f(x − 1) + 3 → right 1, vertical stretch ×2, reflect over the x-axis, then up 3 (apply in that order).
**Dilation check**: 2f(x) doubles every y-value; f(x/2) doubles every x-value (stretches it horizontally).

⚠ f(x + 3) shifts LEFT 3. f(2x) squeezes the graph toward the y-axis by a factor of 1/2 (not by 2).

✓ Factor b first: f(2x − 6) = f(2(x − 3)) → right 3, not 6.

## 1.13–1.14 Function Model Selection & Construction

### Pick a Model (1.13)

| Model | Data clue |
|---|---|
| Linear | Const rate of change |
| Quadratic | Rates change linearly |
| Cubic / higher | Many extrema, inflect. |
| Rational | Asymptote, y ∝ 1/x |
| Piecewise | Rule shifts by interval |

**Geometry clues**: area → quadratic (2 dimensions); volume → cubic (3 dimensions, like a box) in the variable x.
**Inverse variation**: y = k/x; product xy const → rational model.
**Volume example**: 20×20 sheet, cut x-by-x corners & fold → V = x(20 − 2x)², w/ the domain 0 < x < 10 (sides > 0).

### Assumptions & Restrictions

**Domain**: limit to inputs that make sense (time ≥ 0, whole-number items, box sides > 0, etc.) in the given context.
**Range**: outputs must make sense too (no negative counts, lengths, costs or times in the context given).

### Build & Use a Model (1.14)

**From points**: plug each into y = ax² + bx + c → solve the system (or use calculator regression) on the calculator.
**Interpret**: give each parameter & each answer w/ units in context (e.g., "dollars per item sold") or per hour.
**Apply**: predict an output, or solve model = target for the input. Check the answer is in the domain.
**Rational example**: avg cost C(x) = (500 + 2x)/x → HA y = 2 → cost per item → $2 as the number of items x grows.
**Piecewise example**: $5/hr up to 40 hr, then $7.50/hr after → 2 linear pieces joined at x = 40 hours worked.

✓ FRQ 2 (modeling): show the setup, then round to 3 decimal places w/ units in context of the problem.

⚠ Models break outside the data range: flag extrapolation.