The Ultimate AP® Precalculus Unit 2 Cheat Sheet

2.1–2.2 Sequences & Rates of Change

Arithmetic vs. Geometric

	Arithmetic	Geometric
Pattern	Add d	Multiply by r
Formula	$a_n = a_0 + dn$	$g_n = g_0 r^n$
From term k	$a_k + d(n - k)$	$g_k r^{n-k}$
Function type	Linear	Exponential

Common difference $d = a_{n+1} - a_n$. **Common ratio** $r = g_{n+1}/g_n$; constant for every consecutive pair of terms.

From 2 terms: $d = (a_n - a_k)/(n - k)$; $r = (g_n/g_k)^{1/(n-k)}$.

Ex.: $g_2 = 12$, $g_5 = 96 \rightarrow r^3 = 8 \rightarrow r = 2$, $g_0 = 3 \rightarrow g_n = 3 \cdot 2^n$.

Linear vs. Exponential Change

Linear: over equal-length input intervals, output values change by equal DIFFERENCES (additive).

Exponential: over equal-length input intervals, output values change by equal RATIOS (multiplicative).

Rate of change: linear has a constant rate; exponential's rate of change is proportional to its value.

Table check: inputs must be equally spaced. Find successive differences, then successive ratios of the outputs.

Connections

Domain: sequences use whole-number inputs; linear & exponential functions use all real numbers.

Long run: an increasing exponential eventually exceeds any increasing linear or polynomial function.

⚠ Ratios only mean exponential when the inputs are equally spaced. Check the x-column first, every time.

✓ FRQ justify: "Over equal-length input intervals, the outputs change by a constant ratio, so f is exponential."

2.3–2.4 Exponential Functions

General Form $f(x) = ab^x$

Conditions: $a \neq 0$ (initial value, $f(0) = a$), $b > 0$ & $b \neq 1$ (base = growth or decay factor per 1-unit increase in x).

a, b	Behavior	Concavity
$a > 0$, $b > 1$	Increasing	Concave up
$a > 0$, $0 < b < 1$	Decreasing	Concave up
$a < 0$, $b > 1$	Decreasing	Concave down
$a < 0$, $0 < b < 1$	Increasing	Concave down

Key features: domain all reals; horizontal asymptote $y = 0$; always increasing or always decreasing; no extrema.

End behavior ($a > 0$, $b > 1$): as $x \rightarrow \infty$, $f(x) \rightarrow \infty$; as $x \rightarrow -\infty$, $f(x) \rightarrow 0$. The reverse holds for $0 < b < 1$.

Natural base $e \approx 2.718$; $f(x) = e^x$ is used for continuous growth. Shifted form: $f(x) = ab^x + k \rightarrow$ asymptote $y = k$.

Growth factor vs. rate: $b = 1.08$ means 8% growth per unit of x; $b = 0.92$ means 8% decay per unit of x.

Exponent Rules as Transformations

Product: $b^m b^n = b^{m+n} \rightarrow b^{x+k} = b^k \cdot b^x$: a horizontal shift is the same as a vertical dilation by a factor of b^k.

Power: $(b^m)^n = b^{mn} \rightarrow b^{cx} = (b^c)^x$: a horizontal dilation by a factor of 1/c changes the base from b to b^c.

Negative exponent: $b^{-x} = (1/b)^x$, so a reflection over the y-axis turns growth into decay & vice versa.

Roots: $b^{1/k} = \sqrt[k]{b}$. Example: $8^{x/3} = (8^{1/3})^x = 2^x$.

Ex.: $3 \cdot 2^{x+4} = 3 \cdot 16 \cdot 2^x = 48 \cdot 2^x$ (left 4 ≡ stretch by 16).

⚠ $ab^x \neq (ab)^x$: only b gets the exponent.

✓ Symbolic FRQs: rewrite into the form ab^x step by step & name each exponent rule you used along the way.

2.5–2.6 Exponential Modeling & Validation

Building Models

Growth/decay rate r: $f(t) = a(1 + r)^t$; growth → $b = 1 + r$, decay → $b = 1 - r$ (write r as a decimal, e.g., 0.06).

Context	Model
Grows 6% per yr	$a(1.06)^t$
Doubles every 5 hr	$a \cdot 2^{t/5}$
Half-life 30 yr	$a(½)^{t/30}$
Compounded n×/yr	$P(1 + r/n)^{nt}$
Continuous	Pe^{rt}

Equivalent forms: $2^{t/5} = (2^{1/5})^t \approx 1.149^t$ → about 14.9% growth per hour; same function, rewritten form.

Change time units: weekly factor 1.1 → daily factor $1.1^{1/7} \approx 1.014$; yearly factor $1.1^{52} \approx 142$ per year.

From 2 points: (0, 50) & (3, 400) → $a = 50$, $b^3 = 400/50 = 8$ → $b = 2$. Use ratios, then solve for b & a: $y = 50 \cdot 2^x$.

Regression: use exponential regression on a graphing calculator; report a & b to 3 decimal places.

Validating a Model

Residual = actual − predicted. Positive → model underestimates; negative → model overestimates.

Residual plot: random scatter around 0 → model is appropriate. A clear pattern → choose another model.

Choosing a model: equal differences → linear; equal ratios → exponential; changing differences → quadratic.

Assumptions: models are only valid over a reasonable domain; unlimited exponential growth isn't realistic.

⚠ Residual = actual minus predicted, not the other way. The sign decides over vs. underestimate.

✓ Interpret b in context: "the population is multiplied by 1.06 each year" (a 6% annual growth rate).

2.7–2.8 Composition & Inverse Functions

Composition

f(g(x)): output of g becomes input of f. Work inside out: find the inner g(x) first, then apply f to that output value.

Order matters: $f(g(x)) \neq g(f(x))$ in general. $f(x) = x^2$, $g(x) = x + 1 \rightarrow f(g(x)) = (x + 1)^2$ but $g(f(x)) = x^2 + 1$.

From tables or graphs: f(g(2)): read g(2), then find f at that value. Input must be in f's domain to be defined.

Identity $I(x) = x$: $f(I(x)) = I(f(x)) = f(x)$ for every f.

Inverse of a table: swap the input & output columns.

Decomposition: $h(x) = \sqrt{(3x - 1)} = f(g(x))$ w/ $g(x) = 3x - 1$ (inner function) & $f(x) = \sqrt{x}$ (the outer function).

Domain of f(g(x)): x must be in g's domain AND g(x) must be in f's domain (e.g., $\sqrt{(x - 4)}$ needs $x \geq 4$ to be defined).

Transformations as composition: f(x) + k, f(x + h), af(x), f(bx) each compose f w/ a linear function in or out.

Inverse Functions

f^{-1} undoes f: $f(f^{-1}(x)) = f^{-1}(f(x)) = x$; $f(a) = b \rightarrow f^{-1}(b) = a$.

Find it: write $y = f(x)$, swap x & y, solve for y. $f(x) = 2x - 6 \rightarrow x = 2y - 6 \rightarrow f^{-1}(x) = (x + 6)/2 = x/2 + 3$.

Check inverses: verify both $f(g(x)) = x$ & $g(f(x)) = x$; showing only 1 of them isn't enough on an FRQ.

f	f^{-1}
Domain	Range of f
Range	Domain of f
Point (a, b)	Point (b, a)
Graph	Reflect over $y = x$

One-to-one: each output from exactly 1 input (passes horizontal line test). Otherwise restrict the domain.

⚠ $f^{-1}(x) \neq 1/f(x)$. The −1 means the inverse function, not a reciprocal or a negative power of f(x) itself.

2.9–2.10 Logarithmic Expressions

What a Log Means

Definition: $\log_b c = a \Leftrightarrow b^a = c$. A log asks "what power of b gives c?" ($\log_3 81 = 4$ b/c $3^4 = 81$ exactly).

Restrictions: base $b > 0$, $b \neq 1$; argument $c > 0$. $\log_b(0)$ & logs of negatives are undefined (no real value).

Expression	Value	Why
$\log_2 8$	3	$2^3 = 8$
log 1000	3	$10^3 = 1000$
$\log_5 (1/25)$	−2	$5^{-2} = 1/25$
$\log_b 1$	0	$b^0 = 1$
$\log_b b$	1	$b^1 = b$
$\ln e^4$	4	Base e

Common log: $\log x = \log_{10} x$. **Natural log**: $\ln x = \log_e x$.

Estimate: $4 < \log_2 20 < 5$ b/c $2^4 = 16 < 20 < 32 = 2^5$.

Logs Undo Exponentials

Inverses: $y = b^x$ & $y = \log_b x$ undo each other: $\log_b(b^x) = x$ for all real x, & b raised to $\log_b x$ gives x for $x > 0$.

Swap tables: if (2, 9) is on $y = 3^x$, then (9, 2) is on $y = \log_3 x$.

Graphs: $y = \log_b x$ is $y = b^x$ reflected over $y = x$; the horizontal asymptote $y = 0 \leftrightarrow$ vertical asymptote $x = 0$.

Inverse of ab^x: $y = ab^x \rightarrow x = ab^y \rightarrow y = \log_b(x/a)$.

⚠ A log's value is an EXPONENT. $\log_2 8 = 3$, not 8 or 2.

✓ No calculator? Rewrite in exponential form & match powers of the base ($8 = 2^3$, $1/9 = 3^{-2}$).

2.11–2.12 Logarithmic Functions

Graph of $f(x) = a\log_b x$

Feature	$f(x) = \log_b x$
Domain	$x > 0$
Range	All real numbers
Asymptote	Vertical, $x = 0$
x-intercept	(1, 0)
Key point	(b, 1)

$b > 1$, $a > 0$: increasing & concave down. **$0 < b < 1$**: decreasing & concave up. Always 1 direction, never both.

End behavior ($b > 1$): as $x \rightarrow 0^+$, $f(x) \rightarrow -\infty$; as $x \rightarrow \infty$, $f(x) \rightarrow \infty$, but more & more slowly, w/ no horizontal asymptote.

Log Properties

Rule	Statement
Product	$\log_b(xy) = \log_b x + \log_b y$
Quotient	$\log_b(x/y) = \log_b x - \log_b y$
Power	$\log_b(x^n) = n \log_b x$
Change of base	$\log_b x = \ln x / \ln b$

As transformations: $\log_b(cx) = \log_b c + \log_b x \rightarrow$ a horizontal dilation is the same as a vertical shift.

Power rule: $\log_b(x^n) = n \log_b x$, so raising the input to the n is a vertical dilation by a factor of n of the graph.

Any 2 log functions are vertical dilations of each other: $\log_4 x = (1/2) \log_2 x$ b/c $\log_2 4 = 2$ (change of base).

⚠ $\log(x + y) \neq \log x + \log y$, & $\log x / \log y \neq \log(x/y)$. Rules apply to products & quotients INSIDE.

✓ Condense: $2 \log x - \log 5 = \log(x^2/5)$. Expand: $\ln(x^3y) = 3 \ln x + \ln y$. Practice both directions until automatic.

2.13 Exponential & Log Equations

Exponential Equations

Same base: $4^{x+1} = 8^x \rightarrow 2^{2x+2} = 2^{3x} \rightarrow 2x + 2 = 3x \rightarrow x = 2$.

Take a log: $5 \cdot 3^x = 40 \rightarrow 3^x = 8 \rightarrow x = \log_3 8 = \ln 8 / \ln 3 \approx 1.893$ (change of base on a calculator).

Isolate first: divide by a & subtract k before taking the log; $\log(ab^x) \neq x \log(ab)$, only $\log a + x \log b$.

Base e: $2e^{0.4t} = 18 \rightarrow e^{0.4t} = 9 \rightarrow t = \ln 9 / 0.4 \approx 5.49$.

Switch forms: $e^x = 5 \Leftrightarrow x = \ln 5$; $\ln x = 2 \Leftrightarrow x = e^2 \approx 7.39$.

Common error: $\log_2 x = 5$ means $x = 2^5 = 32$, not 5/log 2.

Logarithmic Equations

Rewrite: $\log_2(x - 3) = 4 \rightarrow x - 3 = 2^4 = 16 \rightarrow x = 19$.

Combine first: $\log x + \log(x - 3) = 1 \rightarrow \log(x(x - 3)) = 1 \rightarrow x^2 - 3x = 10 \rightarrow x = 5$ or $x = -2$ (check both).

Check the domain: $x = -2$ makes $\log x$ undefined, so the only solution is $x = 5$ ($x = -2$ is an extraneous root).

Log = log: $\log_b M = \log_b N \rightarrow M = N$ (b/c log is one-to-one), then check that M, N > 0 in the original.

Inequalities & Inverses

Inequalities: base $b > 1$ keeps the sign; $0 < b < 1$ flips it. $(½)^x > 8 \rightarrow 2^{-x} > 2^3 \rightarrow -x > 3 \rightarrow x < -3$ (sign flips).

Graph to check: solutions of $f(x) = c$ are the x-values where $y = f(x)$ & $y = c$ intersect on the graphing calculator.

Inverse of log: $f(x) = \ln(x - 2) + 1 \rightarrow x = \ln(y - 2) + 1 \rightarrow f^{-1}(x) = e^{x-1} + 2$ (f's domain $x > 2 = f^{-1}$'s range).

⚠ Always plug log solutions back in. Any value making a log's argument ≤ 0 is extraneous & must be rejected.

✓ No-calculator FRQ: give exact answers like ln 9 / 0.4; don't round unless the question asks.

2.14–2.15 Log Modeling & Semi-log Plots

Logarithmic Models

When to use: output grows quickly at first, then increases more & more slowly (concave down, no asymptote).

Form: $f(x) = a + b \ln x$ (calculator log regression). Output ↑ by the same amount each time input is ×'d.

Log scales: $pH = -\log[H^+]$ & Richter: each step of 1 = ×10; decibels: each +10 dB = ×10 sound intensity.

Example: pH 3 vs. pH 5 → $10^2 = 100$ times more acidic.

Log model example: $f(x) = 2 + 3 \ln x$ → each time x is multiplied by e, the output f(x) rises by 3.

Semi-log Plots

Semi-log plot: y-axis on a log scale (1, 10, 100, 1000: equal spacing = ×10), x-axis on a normal linear scale.

Key fact: exponential data $y = ab^x$ appears as a straight LINE on a semi-log plot (log y vs. x), not a curve.

Linear model	Means
$\log y = (\log b)x + \log a$	Line in x & log y
Slope	log b
Vertical intercept	log a

Back out the model: slope 0.3 & intercept 1 (base 10) → $b = 10^{0.3} \approx 2$, $a = 10^1 = 10 \rightarrow y \approx 10 \cdot 2^x$ as the model.

Linearize data: take log of each y-value; if (x, log y) is linear, an exponential model fits the original data.

Picking a Model

Compare: exponential → ratios constant; logarithmic → equal outputs for multiplied inputs; check residuals.

⚠ On a semi-log plot, the slope is log b, not b. Undo the log ($b = 10^{\text{slope}}$) to get the growth factor.

✓ Curve on a semi-log plot → data isn't exponential. Straight line → an exponential model fits.