# The Simple AP® Precalculus Unit 3 Cheat Sheet



## 3.1–3.2 Periodic Phenomena & Trig Ratios

### Periodic Functions

**Periodic** (repeating): $f(\theta + k) = f(\theta)$ for all $\theta$; smallest such $k$ = **period**. Output values, rates of change & concavity all repeat every period.
**Use the period**: know one cycle → know all. E.g., period 12 & $f(3) = 5$ → $f(15) = f(27) = 5$.
**Contexts**: tides, Ferris wheels, daylight hours, sound waves → values repeat on a fixed cycle.

### Angles & Radians

**Standard position**: vertex at origin, start on +x-axis; counterclockwise = +, clockwise = −.
**Radian** = arc length ÷ radius ($s = r\theta$). 1 full turn = $2\pi$ rad = 360°. **Convert**: deg × $\pi/180$ = rad.

| Degrees | Radians | Ray on |
|---|---|---|
| 90° | $\pi/2$ | +y-axis |
| 180° | $\pi$ | −x-axis |
| 270° | $3\pi/2$ | −y-axis |
| 360° | $2\pi$ | +x-axis |

**Coterminal** angles: $\theta \pm 2\pi n$ → same terminal ray, same trig values. E.g., $13\pi/6$ & $\pi/6$.

### Sin, Cos, Tan Defined

Point (x, y) on terminal ray, $r = \sqrt{x^2 + y^2}$: $\sin\theta = y/r$, $\cos\theta = x/r$, $\tan\theta = y/x$ ($x \neq 0$).
**Unit circle** (radius 1): point = ($\cos\theta$, $\sin\theta$). $\tan\theta = \sin\theta / \cos\theta$ = slope of the terminal ray.

⚠ Use radian mode on the calculator. Angles are in radians unless a ° sign is shown.

✓ SOH-CAH-TOA matches these defs for acute angles; the x/r, y/r defs work for ANY angle.

## 3.3 Sine & Cosine Values

### Unit Circle Values

| $\theta$ | $\sin\theta$ | $\cos\theta$ | $\tan\theta$ |
|---|---|---|---|
| 0 | 0 | 1 | 0 |
| $\pi/6$ | 1/2 | $\sqrt{3}/2$ | $\sqrt{3}/3$ |
| $\pi/4$ | $\sqrt{2}/2$ | $\sqrt{2}/2$ | 1 |
| $\pi/3$ | $\sqrt{3}/2$ | 1/2 | $\sqrt{3}$ |
| $\pi/2$ | 1 | 0 | undef. |

**Pattern**: sin = $\sqrt{0}/2, \sqrt{1}/2, \sqrt{2}/2, \sqrt{3}/2, \sqrt{4}/2$ as $\theta$ goes 0 → $\pi/2$; cos = same list reversed.

### Reference Angles & Signs

**Reference angle** = acute angle between terminal ray & x-axis. Same size value; the quadrant sets the sign. E.g. $5\pi/3$ → ref angle $\pi/3$ (Q IV).

| Quad. | sin | cos | tan |
|---|---|---|---|
| I | + | + | + |
| II | + | − | − |
| III | − | − | + |
| IV | − | + | − |

**E.g.** $\sin(7\pi/6)$: ref angle $\pi/6$, Q III → $-1/2$. $\cos(5\pi/4) = -\sqrt{2}/2$. $\tan(2\pi/3) = -\sqrt{3}$.

### Symmetry

**$\sin(-\theta) = -\sin\theta$** (odd fn), **$\cos(-\theta) = \cos\theta$** (even fn). $\sin(\pi - \theta) = \sin\theta$, $\cos(\pi - \theta) = -\cos\theta$.

⚠ Rebuild values w/ special triangles: 30-60-90 (sides $1 : \sqrt{3} : 2$) & 45-45-90 (sides $1 : 1 : \sqrt{2}$).

⚠ MCQ Part A & FRQ 3–4 are no-calculator: know exact unit-circle values cold.

## 3.4–3.5 Sine & Cosine Graphs

### Parent Graphs

| Feature | sin θ | cos θ |
|---|---|---|
| Range | $[-1, 1]$ | $[-1, 1]$ |
| Period | $2\pi$ | $2\pi$ |
| Value at 0 | 0 | 1 |
| Zeros | $n\pi$ | $\pi/2 + n\pi$ |
| Max of 1 at | $\pi/2 + 2\pi n$ | $2\pi n$ |
| Inc. on | $(-\pi/2, \pi/2)$ | $(\pi, 2\pi)$ |

**$\cos\theta = \sin(\theta + \pi/2)$**: cosine = sine shifted $\pi/2$ left; both are **sinusoidal** (wave-shaped).

### Key Features

**Amplitude** = (max − min)/2. **Midline**: y = (max + min)/2. **Period** = $2\pi/|b|$. **Frequency** = 1/period.
**Concavity**: sin θ concave down on $(0, \pi)$, concave up on $(\pi, 2\pi)$. **Inflection pts** on the midline.
**Rate of change**: fastest at midline crossings, 0 at max & min. sin θ increasing on $(-\pi/2, \pi/2)$.
**Inc. & concave down** → rate ↓ (slowing near a max). **Inc. & concave up** → rate ↑ (speeding up).
**Quarter-period**: max → midline → min → midline → max, each step ¼ period. Period 8 → step 2.
**Avg. rate of change** over one full period = 0, b/c the start & end output values match.

✓ Sketch 5 pts per cycle: start, ¼, ½, ¾, end. Mark max, min & midline first, then join the pts w/ smooth curves, never sharp corners.

⚠ Midline crossings are NOT zeros unless the midline is y = 0. Zeros exist only if $|d| \leq |a|$.

## 3.6–3.7 Transformations & Modeling

### General Form

**$f(\theta) = a\sin(b(\theta + c)) + d$** (or cos). Each letter controls one feature of the graph:

| Param. | Effect | Note |
|---|---|---|
| a | Vert. stretch | $\|a\|$ = amplitude |
| b | Horiz. stretch | period $2\pi/\|b\|$ |
| c | Phase shift | shift = $-c$ |
| d | Vert. shift | midline y = d |

**Factor b first**: $\sin(2\theta + \pi) = \sin(2(\theta + \pi/2))$ → shift $\pi/2$ left, not $\pi$. Shift = −(inside constant) ÷ b.

### Building a Model

**Steps**: $d = (\max + \min)/2$ → $|a| = (\max - \min)/2$ → $b = 2\pi/\text{period}$ → pick c from a known max or min.
**Max first** → use cos, a > 0. **Midline rising first** → use sin, a > 0. **Min first** → cos, a < 0.
**E.g.** Tide: max 9 m at 2 h, min 3 m at 8 h → d = 6, a = 3, $b = 2\pi/12 = \pi/6$: $h(t) = 3\cos(\pi/6\,(t - 2)) + 6$.
**Ferris wheel**: diam. 40 m, center 25 m, 1 turn/8 min, start low → $h(t) = -20\cos(\pi/4 \cdot t) + 25$.
**Regression** (calculator) for many data pts; check residuals for patterns; give answers w/ units.
**Same curve, other form**: $3\cos(\pi/6\,(t - 2)) + 6 = 3\sin(\pi/6\,(t + 1)) + 6$, b/c $\cos\theta = \sin(\theta + \pi/2)$.
**Units**: period & shift use input units (h, days); a & d use output units (m, °F). $b = 2\pi \div$ period.

✓ FRQ 3 (no calc): find a, b, c, d from data, then answer inc./dec. & concavity Qs.

⚠ Period = time from one max to the next (or 2 × max-to-min time). Midline = the average.

## 3.8–3.9 Tangent & Inverse Trig

### Tangent Function

**$\tan\theta$** = $\sin\theta / \cos\theta$. Period **$\pi$**; vertical asymptotes where $\cos\theta = 0$: $\theta = \pi/2 + n\pi$, n any integer.
**Zeros** at $\theta = n\pi$. Always increasing between asymptotes; range = all reals, so no amplitude.
**Transformed**: $f(\theta) = a\tan(b(\theta + c)) + d$ → period $\pi/|b|$; a, c, d act as for sinusoids.
**Asymptotes of tan(bθ)**: $b\theta = \pi/2 + n\pi$. E.g. $\tan(2\theta)$ has asymptotes at $\theta = \pi/4 + n\pi/2$ & period $\pi/2$.
**Concavity of tan**: concave down left of each zero, concave up right of it (inflection pt at each zero).

### Inverse Trig Functions

**Restrict the domain** so the fn is one-to-one, then invert: input (angle) ↔ output (ratio).

| Fn | Domain | Range |
|---|---|---|
| arcsin x | $[-1, 1]$ | $[-\pi/2, \pi/2]$ |
| arccos x | $[-1, 1]$ | $[0, \pi]$ |
| arctan x | all reals | $(-\pi/2, \pi/2)$ |

**$\sin^{-1}x$** = arcsin x ≠ 1/sin x. **E.g.** $\arcsin(-1/2) = -\pi/6$; $\arccos(-1/2) = 2\pi/3$; $\arctan(-1) = -\pi/4$.
**Composition**: $\arcsin(\sin\theta) = \theta$ only if θ is in $[-\pi/2, \pi/2]$. E.g. $\arcsin(\sin(5\pi/6)) = \pi/6$, not $5\pi/6$.
**Triangle trick**: $\cos(\arctan(3/4))$ → opp 3, adj 4, hyp 5 → cos = 4/5 (+ b/c arctan gives Q I).

⚠ Inverse trig returns ONE angle. Trig eqs have ∞ many sols: use symmetry & period for the rest.

✓ arctan x has horizontal asymptotes $y = \pm\pi/2$; arcsin & arccos have endpoints instead.

## 3.10–3.11 Equations & Reciprocal Fns

### Solving Trig Equations

**Steps**: isolate the trig fn → reference angle → all angles in $[0, 2\pi)$ by quadrant → + period·n.
**E.g.** $2\sin\theta + 1 = 0$ → $\sin\theta = -1/2$ → ref $\pi/6$ in Q III & Q IV ($\sin < 0$) → $\theta = 7\pi/6, 11\pi/6$ (+ $2\pi n$).
**Quadratic form**: $2\cos^2\theta - \cos\theta - 1 = 0$ → $(2\cos\theta + 1)(\cos\theta - 1) = 0$ → $\theta = 0, 2\pi/3, 4\pi/3$.
**No solution**: $\sin\theta = 2$ → no sol, b/c sin & cos stay in $[-1, 1]$. tan θ can equal any real number.
**Check sols**: plug back in; drop any that make a denominator 0 (e.g. tan, sec at $\pi/2$).
**Multiple angle**: $\sin 2\theta = 1$ on $[0, 2\pi)$ → $2\theta = \pi/2, 5\pi/2$ → $\theta = \pi/4, 5\pi/4$. Widen interval for 2θ.

### Inequalities

**Solve the equation first**, then test intervals. E.g. $\cos\theta > 1/2$ on $[0, 2\pi)$ → $[0, \pi/3) \cup (5\pi/3, 2\pi)$.
**Context Qs**: when is h(t) > 7? Solve h(t) = 7, then read intervals off the graph; it repeats.

### Reciprocal Functions

| Fn | Def. | Asymptotes |
|---|---|---|
| csc θ | $1/\sin\theta$ | $\theta = n\pi$ |
| sec θ | $1/\cos\theta$ | $\theta = \pi/2 + n\pi$ |
| cot θ | $\cos\theta/\sin\theta$ | $\theta = n\pi$ |

**Reciprocal graphs**: csc & sec have range $(-\infty, -1] \cup [1, \infty)$ & period $2\pi$; cot has period $\pi$.

⚠ A calculator gives just one angle; find the others in $[0, 2\pi)$ w/ ref angles & signs.

⚠ Don't divide by a trig fn: $\sin\theta\cos\theta = \sin\theta$ → factor $\sin\theta(\cos\theta - 1) = 0$, or you lose $\sin\theta = 0$.

## 3.12 Equivalent Representations

### Pythagorean Identities

**$\sin^2\theta + \cos^2\theta = 1$**. ÷ $\cos^2\theta$ → **$1 + \tan^2\theta = \sec^2\theta$**. ÷ $\sin^2\theta$ → **$1 + \cot^2\theta = \csc^2\theta$**. So $\cos^2\theta = 1 - \sin^2\theta$.
**Use**: $\sin\theta = 3/5$, θ in Q II → $\cos\theta = -4/5$, $\tan\theta = -3/4$. The quadrant picks the sign of the √.

### Sum & Difference

| Expression | Equals |
|---|---|
| $\sin(\alpha + \beta)$ | $\sin\alpha\cos\beta + \cos\alpha\sin\beta$ |
| $\sin(\alpha - \beta)$ | $\sin\alpha\cos\beta - \cos\alpha\sin\beta$ |
| $\cos(\alpha + \beta)$ | $\cos\alpha\cos\beta - \sin\alpha\sin\beta$ |
| $\cos(\alpha - \beta)$ | $\cos\alpha\cos\beta + \sin\alpha\sin\beta$ |

**E.g.** $\sin(\pi/12) = \sin(\pi/3 - \pi/4) = (\sqrt{3}/2)(\sqrt{2}/2) - (1/2)(\sqrt{2}/2) = (\sqrt{6} - \sqrt{2})/4 \approx 0.2588$.

### Double-Angle & Shifts

**$\sin 2\theta = 2\sin\theta\cos\theta$**. **$\cos 2\theta$** = $\cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$. Pick the one that fits.
**Shifts as identities**: $\sin(\theta + \pi/2) = \cos\theta$; $\cos(\theta - \pi/2) = \sin\theta$; $\sin(\theta + \pi) = -\sin\theta$.
**To one fn**: $\sin^2\theta - \cos\theta - 1 = 0$ → $-\cos^2\theta - \cos\theta = 0$ → $-\cos\theta(\cos\theta + 1) = 0$ → $\theta = \pi/2, \pi, 3\pi/2$.
**Simplify**: $(1 - \cos^2\theta)/\sin\theta = \sin^2\theta/\sin\theta = \sin\theta$ ($\sin\theta \neq 0$). Also $\sec\theta\cos\theta = 1$ & $\tan\theta\cos\theta = \sin\theta$.
**Solve**: $\cos 2\theta = \sin\theta$ → $1 - 2\sin^2\theta = \sin\theta$ → $(2\sin\theta - 1)(\sin\theta + 1) = 0$ → $\theta = \pi/6, 5\pi/6, 3\pi/2$.
**Strategy**: work one side; convert to sin & cos; factor; combine fracs; swap via Pythagorean.

✓ FRQ 4: rewrite to solve an eq or find a domain; name the identity you use at each step.

⚠ $\sin(\alpha + \beta) \neq \sin\alpha + \sin\beta$, and $\sin 2\theta \neq 2\sin\theta$.

## 3.13–3.15 Polar Coordinates & Graphs

### Polar Coordinates

**$(r, \theta)$**: distance r from origin, angle θ. **$x = r\cos\theta$**, **$y = r\sin\theta$**, **$r^2 = x^2 + y^2$** & $\tan\theta = y/x$ ($x \neq 0$).
**Negative r** → plot in the opposite direction: $(-r, \theta) = (r, \theta + \pi)$. E.g. $(-2, \pi/6) = (2, 7\pi/6)$.
**Complex no.**: $a + bi = r(\cos\theta + i\sin\theta)$, $r = \sqrt{a^2 + b^2}$. E.g. $1 + i = \sqrt{2}(\cos\pi/4 + i\sin\pi/4)$.

### Polar Graphs

| Form | Graph | Key fact |
|---|---|---|
| $r = a$ | Circle | radius $\|a\|$ |
| $r = a\cos\theta$ | Circle | diameter $\|a\|$ |
| $r = a\sin(n\theta)$ | Rose | n odd: n petals |
| (or $a\cos(n\theta)$) | Rose | n even: 2n |
| $r = a \pm b\sin\theta$ | Limaçon | $\|a\| < \|b\|$: loop |
| (or $\pm b\cos\theta$) | Cardioid | if $\|a\| = \|b\|$ |

**Graph r vs. θ** on x-y axes first, then trace in the polar plane; r = 0 → the curve hits the origin.

### Rates of Change

**r > 0 & increasing** or **r < 0 & decreasing** → distance from origin increasing.
**E.g.** $r = 1 + 2\cos\theta$ on $(0, \pi/2)$: r > 0 & decreasing → the point moves toward the origin (r: 3 → 1).
**Avg. rate of change** of r on $[\theta_1, \theta_2]$ = $(r(\theta_2) - r(\theta_1))/(\theta_2 - \theta_1)$: change in r per radian.

⚠ Distance from origin = $|r|$, not r. r decreasing while negative → point moves AWAY.

✓ Rectangular → polar: sub $x = r\cos\theta$, $y = r\sin\theta$. E.g. $x^2 + y^2 = 4x$ → $r = 4\cos\theta$.