The Ultimate AP® Precalculus Unit 3 Cheat Sheet

3.1–3.2 Periodic Phenomena & Sin, Cos, Tan

Periodic Functions (3.1)

Periodic: f(x + P) = f(x) for all x; **period** P = smallest positive horizontal length of one complete cycle of outputs.

Repeating rates: ↑/↓ intervals, concavity, max/min and avg rates of change all repeat every P units.

Predict: know 1 cycle → know all; f(x + kP) = f(x) for any integer k, e.g., f(50) = f(2) when P = 12.

Examples: tides, daylight hours, Ferris wheel height, sound waves, seasonal temps; each repeats on a fixed period.

Angles & Radians (3.2)

Standard position: vertex at origin, initial ray on +x-axis; CCW turn = +θ, CW turn = −θ (e.g., −π/2 = down).

Radian: θ = s/r (arc length ÷ radius); 2π rad = 360°, π rad = 180°; deg → rad: ×π/180, rad → deg: ×180/π.

Coterminal: θ ± 2πk share one terminal ray → same sin, cos, tan values; e.g., 13π/6 and −11π/6 ↔ π/6.

Arc length: s = rθ (θ in radians); e.g., r = 4, θ = π/3 → s = 4π/3; on the unit circle, arc length = θ itself.

Definitions on Circle of Radius r

Ratio	Circle (x, y, r)	Right △
sin θ	y / r	opp / hyp
cos θ	x / r	adj / hyp
tan θ	y / x (x ≠ 0)	opp / adj

Unit circle (r = 1): terminal point = (cos θ, sin θ); **tan θ** = sin θ/cos θ = slope of terminal ray through origin.

Scaling: on radius r, terminal point = (r cos θ, r sin θ); ratios depend on θ only, e.g., r = 5, θ = π/2 → (0, 5).

✓ Any sin/cos/tan Q: sketch the terminal ray, drop a ⊥ to the x-axis, then read the signs from x and y.

⚠ Calculator in RADIAN mode on MCQ Part B and FRQ 1–2; degree mode gives wrong answers w/o any warning.

3.3 Sine & Cosine Values

Special Angles (Memorize)

θ	sin θ	cos θ	tan θ
0	0	1	0
π/6 (30°)	1/2	√3/2	√3/3
π/4 (45°)	√2/2	√2/2	1
π/3 (60°)	√3/2	1/2	√3
π/2 (90°)	1	0	undef.
π (180°)	0	−1	0
3π/2	−1	0	undef.

Pattern: sin climbs √0/2, √1/2, √2/2, √3/2, √4/2 as θ goes 0 → π/2; cos runs the same list backward.

Reference Angles & Quadrants

Reference angle θ′: acute ∠ between terminal ray and x-axis; |sin θ| = sin θ′ and |cos θ| = cos θ′ for every θ.

Find θ′: QII π − θ, QIII θ − π, QIV 2π − θ; e.g., 5π/6 → π/6, 5π/4 → π/4, 5π/3 → π/3, 11π/6 → π/6, 2π/3 → π/3.

Signs (All Students Take Calc): QI all +, QII only sin +, QIII only tan +, QIV only cos +; csc, sec, cot copy them.

Worked: cos(4π/3): θ′ = π/3, ray in QIII → x < 0, y < 0 → cos(4π/3) = −1/2, sin(4π/3) = −√3/2, tan(4π/3) = √3.

Coordinates: each special angle's point is (cos θ, sin θ), e.g., 3π/4 → (−√2/2, √2/2), 7π/6 → (−√3/2, −1/2).

Symmetry Shortcuts

Odd/even: sin(−θ) = −sin θ, cos(−θ) = cos θ, tan(−θ) = −tan θ; e.g., sin(−π/6) = −1/2, cos(−π/3) = 1/2.

Supplements: sin(π − θ) = sin θ, cos(π − θ) = −cos θ; **half-turn**: θ + π flips the sign of both sin and cos.

⚠ Part A is no-calc: exact values like √3/2 must be instant. A decimal like 0.866 won't match the choices.

✓ Ray in QII, QIII or QIV? Find θ′ first, get the value from the table, then attach the quadrant's sign.

3.4–3.5 Sine & Cosine Graphs, Sinusoids

Parent Graphs (3.4)

Feature	y = sin θ	y = cos θ
At θ = 0	0, rising	1 (max)
Period	2π	2π
Range	[−1, 1]	[−1, 1]
Symmetry	odd (origin)	even (y-axis)
Zeros	θ = kπ	θ = π/2 + kπ

Unit-circle link: sin θ = y-coord, cos θ = x-coord of the terminal point as it travels once around the circle every 2π.

Rate of change: steepest where graph crosses midline; ≈ 0 near max/min, where the curve flattens out.

Concavity: concave ↓ above the midline, concave ↑ below it; inflection points = midline crossings.

Extrema: sin max at π/2 + 2πk; cos max at 2πk.

Monotonic: sin ↑ on (−π/2, π/2), ↓ on (π/2, 3π/2); cos ↓ on (0, π), ↑ on (π, 2π); avg rate over 1 full period = 0.

Sinusoidal Functions (3.5)

Sinusoid: f(θ) = a sin(b(θ + c)) + d or a cos(b(θ + c)) + d; cos is a shifted sine: cos θ = sin(θ + π/2).

Amplitude = |a| = (max − min)/2, the vertical distance from the midline up to max or down to min.

Midline: y = d = (max + min)/2; **period** = 2π/|b|; **frequency** = |b|/2π = 1/period = cycles per unit of input.

Quarter-cycle rule: max, midline, min, midline, max sit period/4 apart; plot these 5, then sketch.

Read a graph: amplitude = half the max-to-min height; period = horizontal distance between 2 consecutive peaks.

⚠ Amplitude is never negative. a = −3 → amplitude 3, plus a reflection of the graph over its midline.

✓ Asked if f is ↑/↓ or concave ↑/↓ on an interval? Locate the interval vs. the midline and the extrema.

3.6–3.7 Transformations & Modeling

f(θ) = a sin(b(θ + c)) + d (3.6)

Param.	Effect	Reads as
a	vertical dilation	amp. = |a|
a < 0	reflect in midline	max ↔ min
b	horiz. dilation	period 2π/|b|
c	phase shift	left c if c > 0
d	vert. translation	midline y = d

Factor b first: sin(2θ − π) = sin(2(θ − π/2)) → shift right π/2, not π; period = 2π/2 = π, amplitude still 1.

Horizontal dilation by factor 1/|b|: b = 3 squeezes the graph 3×, so period = 2π/3; b = 1/2 stretches it to 4π.

Building a Model From Data (3.7)

Step 1: midline d = (max + min)/2 and |a| = (max − min)/2, using the context's high and low values.

Step 2: period = time from one max to the next (or 2× the time from a max to the next min); then solve b = 2π/period.

Step 3: use cos if t = 0 is a max (a > 0) or min (a < 0); use sin if t = 0 is on the midline, rising (a > 0) or falling (a < 0).

Step 4: phase shift c = horizontal offset to the 1st max (for cos) or 1st rising midline point (for sin).

Worked: wheel 2 m to 42 m high, 1 lap per 8 min, starts at bottom: a = −20, d = 22 → h(t) = −20 cos(πt/4) + 22.

Regression & Data Checks

Sinusoidal regression on calc for noisy data; a good model's residual plot shows no pattern, just random scatter.

Combined models: sinusoid + linear trends up while oscillating; sinusoid × decaying exponential = damped wave.

Interpret: h(5) = height at t = 5 min; avg rate (h(5) − h(1))/4 = m per min; give units in every FRQ answer.

✓ FRQ 3: label each key point (max, min, midline) w/ its t-value; justify period, amplitude and midline.

3.8–3.9 Tangent & Inverse Trig

Tangent Function (3.8)

tan θ = sin θ/cos θ = slope of the terminal ray; undefined wherever cos θ = 0 (vertical ray at π/2, 3π/2).

Graph: period π, range all reals, zeros at θ = kπ, vertical asymptotes at θ = π/2 + kπ, passes through (π/4, 1).

Behavior: ↑ on every branch; concave ↓ on (−π/2, 0), concave ↑ on (0, π/2); odd, so origin symmetry.

Transformed: a tan(b(θ + c)) + d → period π/|b|, no amplitude; asymptotes shift w/ c, stretch w/ b.

Graph of $\tan^{-1}$: reflect the middle tan branch over y = x → horizontal asymptotes y = ±π/2, always ↑.

Inverse Trig Functions (3.9)

Function	Domain	Range (output)
$\sin^{-1} x$	[−1, 1]	[−π/2, π/2]
$\cos^{-1} x$	[−1, 1]	[0, π]
$\tan^{-1} x$	(−∞, ∞)	(−π/2, π/2)

Why restricted: sin, cos, tan aren't 1-to-1, so each domain is cut to one ↑ or ↓ piece that hits every output once.

Meaning: $\sin^{-1}(x)$ = the one angle in [−π/2, π/2] whose sine is x; the input is a ratio, the output is an angle.

Worked: $\cos^{-1}(-1/2) = 2\pi/3$ (in [0, π]), not 4π/3; $\tan^{-1}(-1) = -\pi/4$, not 3π/4; $\sin^{-1}(1) = \pi/2$, $\cos^{-1}(0) = \pi/2$.

Compositions: $\sin(\sin^{-1} x) = x$ on [−1, 1], but $\sin^{-1}(\sin(5\pi/6)) = \pi/6$, b/c 5π/6 is outside [−π/2, π/2].

Calc use (Part B): $\sin^{-1}(0.3) \approx 0.305$ gives only the QI answer; the QII answer is π − 0.305 ≈ 2.837.

⚠ $\sin^{-1} x$ means arcsin x, NOT 1/sin x. The reciprocal 1/sin x is a different function: csc x = $(\sin x)^{-1}$.

⚠ An inverse trig output must land in its range: $\cos^{-1}$ never gives a negative angle; $\sin^{-1}$ and $\tan^{-1}$ never pass ±π/2.

3.10–3.11 Trig Equations & sec, csc, cot

Solving Equations (3.10)

Isolate the trig function, find the reference angle θ′, then list every angle in [0, 2π) with the right sign.

Worked: 2 sin θ + 1 = 0 → sin θ = −1/2 → θ′ = π/6; sin < 0 in QIII, QIV → θ = π + π/6 = 7π/6 and 2π − π/6 = 11π/6.

All solutions: add + 2πk to each sin/cos answer and + πk to a tan answer, where k is any integer; e.g., θ = 7π/6 + 2πk.

Inside b: sin(2θ) = 1 → 2θ = π/2 + 2πk → θ = π/4 + πk; solve for 2θ first, then divide; on [0, 2π): π/4, 5π/4.

Factor trig quadratics: 2cos²θ − cos θ − 1 = 0 → (2cos θ + 1)(cos θ − 1) = 0 → cos θ = −1/2 or 1 → θ = 0, 2π/3, 4π/3.

Calc check (Part B): graph y = left side and y = right side in radian mode, then find every intersection in the window.

Inequalities

Method: solve the = case, mark those angles on [0, 2π), then test 1 point in each interval on the graph.

Worked: cos θ > 1/2 on [0, 2π) → cos θ = 1/2 at π/3, 5π/3; cos sits above 1/2 near 0 and 2π → [0, π/3) ∪ (5π/3, 2π).

Reciprocal Functions (3.11)

Func.	Equals	Asymptotes
csc θ	1 / sin θ	θ = kπ
sec θ	1 / cos θ	θ = π/2 + kπ
cot θ	cos θ / sin θ	θ = kπ

Graphs: csc, sec period 2π, range (−∞, −1] ∪ [1, ∞), U-shapes touch sin/cos at ±1; cot period π, ↓.

Values: sec(π/3) = 2, csc(π/6) = 2, cot(π/4) = 1; sec, csc never fall between −1 and 1, b/c |sin|, |cos| ≤ 1.

⚠ Don't divide both sides by sin θ or cos θ: you lose every solution where it equals 0. Factor it out.

✓ Q gives a restricted domain? Check every solution lies inside it before you write the final answer.

3.12 Equivalent Representations

Pythagorean Identities

Core: sin²θ + cos²θ = 1 for every θ, b/c x² + y² = 1 on the unit circle; rearranged: sin²θ = 1 − cos²θ, cos²θ = 1 − sin²θ.

÷ cos²θ: 1 + tan²θ = sec²θ. **÷ sin²θ**: 1 + cot²θ = csc²θ.

Find one from another: sin θ = 3/5 in QII → cos θ = −√(1 − 9/25) = −4/5 (QII: x < 0) → tan θ = −3/4, sec θ = −5/4.

Sum & Double-Angle Identities

Identity	Expansion
sin(α + β)	sin α cos β + cos α sin β
sin(α − β)	sin α cos β − cos α sin β
cos(α + β)	cos α cos β − sin α sin β
cos(α − β)	cos α cos β + sin α sin β
sin 2θ	2 sin θ cos θ
cos 2θ	cos²θ − sin²θ

cos 2θ forms: also 2cos²θ − 1 or 1 − 2sin²θ; pick the form that leaves only one trig function, then factor or solve.

Worked: sin(π/12) = sin(π/3 − π/4) = (√3/2)(√2/2) − (1/2)(√2/2) = (√6 − √2)/4 ≈ 0.259; check on calc in Part B.

Rewriting Strategy

Convert sec, csc, tan, cot → sin and cos, combine into one fraction, then use sin²θ + cos²θ = 1; e.g., tan θ cos θ = sin θ.

Solve w/ identities: sin 2θ = sin θ → 2 sin θ cos θ − sin θ = 0 → sin θ(2cos θ − 1) = 0 → θ = 0, π/3, π, 5π/3.

Verify an identity: work one side only until it matches the other; plug in θ = π/6 as a quick sanity check.

Common rewrites: (1 − cos²θ)/sin θ = sin θ (θ ≠ kπ); sec θ − cos θ = sin²θ/cos θ = sin θ tan θ; cot θ sin θ = cos θ.

⚠ sin(α + β) ≠ sin α + sin β, and sin 2θ ≠ 2 sin θ. Trig functions never distribute over sums.

✓ FRQ 4: put each identity step on its own line; an equivalent form earns credit only with work shown.

3.13–3.15 Polar Coordinates & Functions

Polar ↔ Rectangular (3.13)

Point (r, θ): r = signed distance from origin, θ = angle from +x-axis; x = r cos θ, y = r sin θ; (2, π/6) → (√3, 1).

Back: $r^2 = x^2 + y^2$, tan θ = y/x; choose θ by the quadrant of (x, y), e.g., (−1, 1) → (√2, 3π/4), not (√2, −π/4).

Not unique: (r, θ) = (r, θ + 2πk) = (−r, θ + π); a negative r plots on the ray opposite θ, e.g., (−2, π/2) = (0, −2).

Complex numbers: a + bi ↔ point (a, b) ↔ r(cos θ + i sin θ), r = √(a² + b²); e.g., 1 + i ↔ (√2, π/4) in polar.

Polar Graphs (3.14)

Equation	Curve	Key fact
r = a	circle	radius |a|
r = a cos θ	circle	diam. |a| on x-axis
r = a cos(nθ)	rose	n odd: n; even: 2n
r = a + b cos θ	limaçon	|a| < |b|: loop
r = a(1 + cos θ)	cardioid	hits origin once

Graph from r vs θ: plot r = f(θ) on rectangular axes, then wrap it around the origin as θ sweeps 0 → 2π.

Worked: r = 2 + 2cos θ: r = 4 at θ = 0, 2 at π/2, 0 at π → cardioid w/ its cusp at the origin, symmetric over x-axis.

Symmetry tests: r(−θ) = r(θ) → symmetric over x-axis (cos curves); r(π − θ) = r(θ) → over y-axis (sin curves).

Rates of Change in Polar (3.15)

Distance from origin ↑ when r > 0 and r ↑, or when r < 0 and r ↓; either way |r| grows. It ↓ whenever |r| shrinks.

Avg rate of r on $[\theta_1, \theta_2]$ = $(r(\theta_2) - r(\theta_1))/(\theta_2 - \theta_1)$, i.e., change in r per radian; its sign says whether r ↑ or ↓.

Extrema: max of |r| = point farthest from origin; r = 0 → curve passes through the pole; on a rose, max |r| = petal tip.

⚠ r < 0 and decreasing means the point moves AWAY from the origin. Judge distance by |r|, not r.