The Simple AP® Precalculus Unit 4 Cheat Sheet

4.1–4.2 Parametric Functions & Planar Motion

Parametric Basics

Parametric function: x(t) & y(t) both depend on input t (often time) → one pt (x, y) per t.

Graph: plot pts in order of ↑t & connect them; arrows show **orientation** (direction of travel).

t	x = t²	y = t − 1	point
−1	1	−2	(1, −2)
0	0	−1	(0, −1)
1	1	0	(1, 0)
2	4	1	(4, 1)

Read it: these pts trace a sideways parabola, x = (y + 1)², from bottom to top as t ↑.

Eliminate t: solve one eqn for t, plug into the other → x–y eqn (shows shape, not direction).

Planar Motion

Position at time t = (x(t), y(t)); x(t) = left/right motion, y(t) = up/down motion.

Moving right when x(t) ↑, **left** when x(t) ↓; **up** when y(t) ↑, **down** when y(t) ↓.

Highest pt: where y(t) is greatest; **rightmost pt**: where x(t) is greatest (often different t's).

Ex.: x = 3t, y = −16t² + 32t → always moving right; up until t = 1, then down; max height 16 ft.

⚠ The graph shows the path, not the timing: read t from the eqns or a t | x | y table, not the picture.

✓ Start w/ a t | x | y table: it shows direction, turnarounds & repeated pts faster than algebra.

4.3–4.4 Rates of Change, Circles & Lines

Rates of Change

Avg rate of x on $[t_1, t_2]$ = Δx/Δt = change in x ÷ change in t; same idea for y (Δy/Δt).

Slope (rate of y w.r.t. x) = Δy/Δx = (Δy/Δt) ÷ (Δx/Δt) = slope of the secant b/w 2 pts.

Ex.: x = t², y = t³ on [1, 2]: Δx/Δt = 3, Δy/Δt = 7 → slope Δy/Δx = 7/3, from (1, 1) to (4, 8).

Δx/Δt = 0: x ends where it started → the secant line is vertical, so Δy/Δx is undefined.

Circles

Path	x(t)	y(t)
Unit, CCW	cos t	sin t
r, ctr (h, k)	h + r cos t	k + r sin t
Clockwise	h + r cos t	k − r sin t

One lap: 0 ≤ t ≤ 2π, starting at the rightmost pt (h + r, k) when t = 0 & going counterclockwise.

Clockwise: use k − r sin t for y. **Faster**: cos(2t) & sin(2t) → 2 laps on [0, 2π], period π.

Lines & Segments

Line through (x_1, y_1) w/ slope b/a: x = x_1 + at, y = y_1 + bt, for all real t (not just 0 to 1).

Segment (x_1, y_1) → (x_2, y_2), 0 ≤ t ≤ 1: x = x_1 + (x_2 − x_1)t, y = y_1 + (y_2 − y_1)t (start + change·t).

Ex.: (1, 2) → (7, −1): x = 1 + 6t, y = 2 − 3t; t = 1/2 → midpoint (4, 0.5); t = 1 → end (7, −1).

⚠ Δy/Δx ≠ Δy/Δt. Divide y's rate by x's rate to get the slope; y's rate alone is just vertical change.

✓ Limit t to trace only part of a curve: segment ↔ 0 ≤ t ≤ 1; upper half circle ↔ 0 ≤ t ≤ π.

4.5–4.6 Implicit Functions & Conic Sections

Implicit Functions

Implicit: one eqn links x & y, e.g. x² + y² = 25 or xy = 6; y may not be a function of x.

Split into functions: y = √(25 − x²) (top half) & y = −√(25 − x²) (bottom half) of the circle.

Points: plug in x, solve for every y. Ex.: x² + y² = 25, x = 3 → y = ±4 → (3, 4) & (3, −4).

Conic Sections (centered at origin)

Conic	Standard form	Key feature
Parabola	y = ax²	vertex (0, 0)
Circle	x² + y² = r²	radius r
Ellipse	x²/a² + y²/b² = 1	semi-axes a, b
Hyperbola	x²/a² − y²/b² = 1	y = ±(b/a)x

Shift: replace x w/ x − h & y w/ y − k → the center (or vertex) moves to (h, k); same shape.

Ellipse: vertices (±a, 0), co-vertices (0, ±b); a > b → wider than tall; a = b → a circle.

Hyperbola: opens left/right if the x² term is +, up/down if y² is +; it has 2 asymptotes.

Identify: x² & y² same sign → circle/ellipse; opposite → hyperbola; one squared → parabola.

Sideways parabola: x = ay² opens right (a > 0) or left; it fails the vertical line test.

⚠ Complete the square first: x² + 6x + y² = 7 → (x + 3)² + y² = 16, center (−3, 0), r = 4.

✓ Sketch: plot the center, step ±a along x & ±b along y, then draw the curve through those pts.

4.7 Parametrizing Implicitly Defined Functions

Standard Parametrizations

Curve	x(t)	y(t)
y = f(x)	t	f(t)
x = g(y)	g(t)	t
x² + y² = r²	r cos t	r sin t
x²/a² + y²/b² = 1	a cos t	b sin t
x²/a² − y²/b² = 1	a sec t	b tan t

Functions

Any y = f(x): use x = t, y = f(t), traced left → right. Ex.: y = x³ − 2 → (t, t³ − 2) for all t.

Sideways x = g(y): use x = g(t), y = t. Ex.: x = y² + 1 → (t² + 1, t), traced bottom → top.

Ellipses & Hyperbolas

Ellipse ctr (h, k): x = h + a cos t, y = k + b sin t, 0 ≤ t ≤ 2π, counterclockwise from (h + a, k).

Why it works: ((x − h)/a)² + ((y − k)/b)² = cos²t + sin²t = 1 for every t (Pythagorean identity).

Ex.: x²/9 + y²/4 = 1 → x = 3cos t, y = 2sin t; t = π/2 → top pt (0, 2); t = π → (−3, 0).

Hyperbola ctr (h, k): x = h + a sec t, y = k + b tan t, b/c the identity sec²t − tan²t = 1.

Not unique: (t, t²) & (2t, 4t²) both trace y = x²; many choices trace one curve at different speeds.

⚠ Use a cos t & b sin t, not a² & b²: a & b are the semi-axis lengths (square roots of denominators).

✓ Check: plug x(t) & y(t) back into the original eqn; it must simplify to a true identity.

4.8 Vectors

Vector Basics

Vector: has magnitude (length) & direction; **v** = ⟨a, b⟩ = a**i** + b**j**, a = x-part, b = y-part.

From P to Q: $\langle x_2 - x_1, y_2 - y_1 \rangle$ (end minus start).

Magnitude |**v**| = √(a² + b²). Ex.: |⟨3, 4⟩| = 5.

Components: **v** = ⟨|**v**| cos θ, |**v**| sin θ⟩, θ = direction angle. Ex.: length 10, θ = 30° → ⟨5√3, 5⟩.

Unit vector (length 1) in the direction of **v**: **v**/|**v**|. Ex.: ⟨3, −4⟩ has length 5 → ⟨3/5, −4/5⟩.

Operations

Operation	Rule	Picture
u + **v**	add components	head-to-tail
k**v**	⟨ka, kb⟩	scale by |k|
u − **v**	**u** + (−**v**)	tip of v → tip of u
u · **v**	$u_1v_1 + u_2v_2$	a scalar

Ex.: ⟨2, 1⟩ + ⟨−5, 3⟩ = ⟨−3, 4⟩; 3⟨2, 1⟩ = ⟨6, 3⟩; −⟨2, 1⟩ = ⟨−2, −1⟩ (k < 0 flips direction).

Dot Product

Dot product: **u** · **v** = $u_1v_1 + u_2v_2$ = |**u**||**v**| cos θ, where θ = the angle b/w the vectors.

Angle: cos θ = (**u** · **v**)/(|**u**||**v**|). Ex.: ⟨1, 0⟩ & ⟨1, 1⟩ → cos θ = 1/(1·√2) = 1/√2 → θ = π/4.

Perpendicular when **u** · **v** = 0. Ex.: ⟨2, 1⟩ · ⟨−1, 2⟩ = −2 + 2 = 0, so they meet at 90°.

⚠ A dot product is a number, not a vector.

✓ Sketch the vectors tail to tail before finding an angle: it catches a θ in the wrong quadrant.

4.9 Vector-Valued Functions

Position Vectors

Vector-valued function: **p**(t) = ⟨x(t), y(t)⟩ gives the position at time t as a vector from the origin.

Distance from origin = |**p**(t)|. Ex.: **p**(2) = ⟨3, 4⟩ → √(9 + 16) = 5 units away at t = 2.

Change & Velocity

Quantity	Formula
Position	**p**(t) = ⟨x(t), y(t)⟩
Displacement	**p**(b) − **p**(a)
Avg velocity	(**p**(b) − **p**(a))/(b − a)
Dist. from origin	|**p**(t)|

Displacement: $\mathbf{p}(t_2) - \mathbf{p}(t_1)$, an arrow from the start pt to the end pt (straight, not the path).

Avg velocity = displacement ÷ Δt = ⟨Δx/Δt, Δy/Δt⟩.

Ex.: **p**(t) = ⟨t², 2t⟩ on [1, 3]: ⟨1, 2⟩ → ⟨9, 6⟩, displacement ⟨8, 4⟩, avg velocity ⟨4, 2⟩.

Motion

Components: x(t) ↑ → moving right; y(t) ↑ → moving up. Read the 2 parts separately.

Circular motion: **p**(t) = ⟨r cos(ωt), r sin(ωt)⟩; ω = angular speed, period 2π/ω, |**p**(t)| = r.

Ex.: r = 4, ω = π/2 → one lap every 4 s; at t = 1, **p** = ⟨0, 4⟩, the top of the circle (¼ lap).

⚠ Displacement ≠ distance: a full lap of a circle has displacement ⟨0, 0⟩ but distance 2πr.

✓ Calc BC preview: derivatives later replace these average rates to give velocity at an instant.

4.10–4.11 Matrices, Determinant & Inverse

Matrix Basics (rows split by ;)

Matrix: m × n = m rows by n columns. [1 2; 3 4] is 2 × 2; [1 2 3] is 1 × 3 (one row).

Add: same size only, entry by entry. **Scalar** kA: multiply each entry by k. Ex.: 2[1 −3] = [2 −6].

Product AB: columns of A must = rows of B; each entry = a row of A · a column of B.

Ex.: [1 2; 3 4][5 6; 7 8] = [19 22; 43 50]; 19 = 1·5 + 2·7 (row 1 · column 1), 22 = 1·6 + 2·8.

Order matters: AB ≠ BA in general. **Identity** I = [1 0; 0 1]: AI = IA = A, like multiplying by 1.

Determinant

det [a b; c d] = ad − bc (main diagonal minus other diagonal). Ex.: det [3 1; 4 2] = 6 − 4 = 2.

Area: |det| = area of the parallelogram made by the 2 column vectors; det = 0 → no area.

Ex.: vectors ⟨3, 0⟩ & ⟨1, 2⟩ → det [3 1; 0 2] = 6, so the parallelogram's area is 6 (base 3 · height 2).

Inverse

Inverse: A^{-1} = (1/det)[d −b; −c a]: swap a & d, negate b & c, divide by det. $AA^{-1} = I$.

Ex.: A = [3 1; 4 2], det 2 → A^{-1} = [1 −1/2; −2 3/2]. If det = 0, there is no inverse at all.

Systems: write AX = B, then X = A^{-1}B; multiply A^{-1} on the left side b/c order matters.

⚠ $(AB)^{-1} = B^{-1}A^{-1}$: reverse the order.

✓ Check sizes first: (2 × 3)(3 × 1) works → 2 × 1; (2 × 3)(2 × 3) is undefined b/c 3 ≠ 2.

4.12–4.14 Linear Transformations & Models

Linear Transformations

Linear transformation: L(x, y) = (ax + by, cx + dy) ↔ matrix [a b; c d]; L(**v**) = A**v** (**v** as a column).

Columns of the matrix = where ⟨1, 0⟩ & ⟨0, 1⟩ land. The origin always stays at the origin.

Transformation	Matrix
Rotate θ CCW	[cos θ −sin θ; sin θ cos θ]
Reflect over x-axis	[1 0; 0 −1]
Reflect over y = x	[0 1; 1 0]
Dilate by k	[k 0; 0 k]

Matrices as Functions

Composition: do B, then A → use AB (the right matrix acts first). Ex.: 2 turns of 90° = 180°.

|det A| = area scale factor; **det = 0** → no inverse (the plane flattens onto a line or a pt).

Inverse map: A^{-1} undoes A; det < 0 → the picture is flipped over (includes a reflection).

Modeling

Transition matrix: probabilities of moving b/w states each step; each state's outflows sum to 1.

Ex.: A = [0.9 0.2; 0.1 0.8], start ⟨100, 0⟩ → ⟨90, 10⟩ → ⟨83, 17⟩ (multiply by A each step).

Steady state: A**x** = **x**; the numbers stop changing.

Past states: multiply by A^{-1} instead.

⚠ AB means do B first: rotate-then-reflect ≠ reflect-then-rotate, so order matters.

✓ Steady state above: 0.1x = 0.2y & x + y = 100 → (66.7, 33.3). Solve Ax = x w/ the total fixed.