The Ultimate AP® Precalculus Unit 4 Cheat Sheet

4.1–4.2 Parametric Functions & Planar Motion

Parametric Basics

Parametric function: f(t) = ⟨x(t), y(t)⟩; one input t (often time) sets both coordinates → one pt in the plane.

Graph: plot (x(t), y(t)) for many t & connect in order of ↑t; arrows show **orientation** (direction of travel).

t	x = t²	y = t − 1	point
−1	1	−2	(1, −2)
0	0	−1	(0, −1)
1	1	0	(1, 0)
2	4	1	(4, 1)

Read it: the table traces x = (y + 1)², a sideways parabola, bottom → top as t ↑; x = 1 at t = ±1 gives 2 different pts.

Eliminate t: solve one eqn for t, substitute into the other → an x–y eqn of the path (shape only; loses direction & timing).

Planar Motion

Position at time t = (x(t), y(t)); x(t) = horizontal motion, y(t) = vertical motion, each its own function of t.

Moving right when x(t) ↑, **left** when x(t) ↓; **up** when y(t) ↑, **down** when y(t) ↓. Check x & y separately.

Extrema: rightmost pt where x(t) has a max; highest pt where y(t) has a max; these rarely happen at the same t.

Turnarounds: x switches ↑↔↓ → object reverses left/right; y switches → reverses up/down (e.g. top of a toss).

Ex.: x = 3t, y = −16t² + 32t → always moving right; up on 0 < t < 1, down after; max height y(1) = 16 ft.

Same Path, Different Motion

Different parametrizations can trace one curve at different speeds, in opposite directions, or more than once.

⚠ A parametric graph shows the path, not the timing: read t from the eqns or a table, never from the picture alone.

✓ Build a t | x | y table first: it shows direction, turnarounds & repeated pts faster than eliminating t.

4.3–4.4 Rates of Change, Circles & Lines

Parametric Rates of Change

Avg rate of x on [t₁, t₂] = (x(t₂) − x(t₁))/(t₂ − t₁); same for y. Each = horizontal or vertical change per unit of t.

Avg rate of y w.r.t. x = Δy/Δx = (Δy/Δt)/(Δx/Δt) = slope of the secant line b/w the two pts on the curve.

Ex.: x = t², y = t³ on [1, 2]: Δx/Δt = 3, Δy/Δt = 7 → Δy/Δx = 7/3, the slope of the secant from (1, 1) to (4, 8).

Δx/Δt = 0 on an interval: x ends where it started → the secant is vertical & Δy/Δx is undefined (not 0, not ∞).

Instantaneous rate of y w.r.t. x at t ≈ Δy/Δx over smaller & smaller intervals around t (slope of the tangent).

Circles

Path	x(t)	y(t)
Unit, CCW	cos t	sin t
r, ctr (h, k)	h + r cos t	k + r sin t
Clockwise	h + r cos t	k − r sin t

One lap: 0 ≤ t ≤ 2π; t on [0, π] = upper half only. **Faster**: cos(2t), sin(2t) → 2 laps on [0, 2π], period π.

Start pt: t = 0 gives (h + r, k), the rightmost pt; swap sin & cos to start at the top (h, k + r) & go clockwise.

Lines & Segments

Line through (x₁, y₁) w/ slope b/a: x = x₁ + at, y = y₁ + bt; a = Δx per unit t, b = Δy per unit t, for all real t.

Segment (x₁, y₁) → (x₂, y₂), 0 ≤ t ≤ 1: x = x₁ + (x₂ − x₁)t, y = y₁ + (y₂ − y₁)t; t = 1/2 → midpoint, t = 1 → end.

Ex.: (1, 2) → (7, −1): x = 1 + 6t, y = 2 − 3t, slope −3/6 = −1/2; t = 1/3 → (3, 1), one third of the way along.

⚠ Δy/Δx ≠ Δy/Δt. Divide y's rate by x's rate to get slope; y's rate alone is only vertical change per unit t.

✓ Restrict t to trace only part of a curve: segment ↔ 0 ≤ t ≤ 1; upper half circle ↔ 0 ≤ t ≤ π; one lap ↔ 0 ≤ t ≤ 2π.

4.5–4.6 Implicit Functions & Conic Sections

Implicitly Defined Functions

Implicit: x & y tied by one eqn, f(x, y) = 0, e.g. x² + y² = 25 or xy = 6; y need not be a function of x at all.

Fails vertical line test? Split into pieces: y = √(25 − x²) (top) & y = −√(25 − x²) (bottom), each a function.

Ex.: xy = 6 → y = 6/x, a function (2 branches, Q1 & Q3); x² + y² = 25 → not a function, so it needs 2 pieces.

Points: plug in x, solve for every y (0, 1, 2+ values). Ex.: x² + y² = 25, x = 3 → y = ±4 → (3, 4) & (3, −4).

Rate of change at a pt: slope of the tangent; + if y ↑ as x ↑ nearby, − if y ↓; vertical tangent where curve turns back in x.

Conic Sections (centered at origin)

Conic	Standard form	Key feature
Parabola	y = ax²	vertex (0, 0)
Circle	x² + y² = r²	radius r
Ellipse	x²/a² + y²/b² = 1	semi-axes a, b
Hyperbola	x²/a² − y²/b² = 1	y = ±(b/a)x

Shift: replace x w/ x − h & y w/ y − k → center or vertex moves to (h, k); shape & axes stay the same.

Ellipse: a > b → wider than tall; vertices (±a, 0), co-vertices (0, ±b); b > a → taller; a = b → circle of radius a.

Hyperbola: x² term + → opens left/right, vertices (±a, 0); y² term + → opens up/down. Asymptotes cross at center.

Sideways parabola x − h = a(y − k)² opens right (a > 0) or left (a < 0); not a function of x, so it is implicit.

Identify the Conic

Squared terms: same sign & coef. → circle; same sign, diff. coef. → ellipse; opposite signs → hyperbola; one → parabola.

⚠ Complete the square before reading features: x² + 6x + y² = 7 → (x + 3)² + y² = 16, center (−3, 0), r = 4.

✓ Sketch from standard form: plot center (h, k), step ±a along x & ±b along y, then draw through those pts.

4.7 Parametrizing Implicitly Defined Functions

Standard Parametrizations

Curve	x(t)	y(t)
y = f(x)	t	f(t)
x = g(y)	g(t)	t
x² + y² = r²	r cos t	r sin t
x²/a² + y²/b² = 1	a cos t	b sin t
x²/a² − y²/b² = 1	a sec t	b tan t

Functions & Parabolas

Any y = f(x): x = t, y = f(t) traces the graph left → right. Ex.: y = x³ − 2 → (t, t³ − 2) for all real t, same graph.

Sideways x = g(y): x = g(t), y = t. Ex.: x = y² + 1 → (t² + 1, t), traced bottom → top as t ↑; vertex at t = 0.

Ellipses & Hyperbolas

Ellipse ctr (h, k): x = h + a cos t, y = k + b sin t, 0 ≤ t ≤ 2π, CCW from (h + a, k), b/c cos²t + sin²t = 1.

Ex.: x²/9 + y²/4 = 1 → x = 3cos t, y = 2sin t; t = π/2 → top pt (0, 2); t = π → left vertex (−3, 0); t = 2π → start.

Hyperbola centered (h, k): x = h + a sec t, y = k + b tan t, b/c sec²t − tan²t = 1 (divide sin² + cos² = 1 by cos²).

Branches: −π/2 < t < π/2 → sec t > 0 → right branch; π/2 < t < 3π/2 → left branch; t ≠ π/2 or 3π/2 (sec undefined).

Opens up/down: (y − k)²/b² − (x − h)²/a² = 1 → y = k + b sec t, x = h + a tan t (sec goes w/ the positive term).

Choices & Checks

Not unique: (t, t²), (2t, 4t²) & (t³, t⁶) all trace y = x²; they differ in speed & in which t gives which pt on the curve.

Direction: replace t w/ −t (or swap sin & cos) to go clockwise; shrink the t-interval to trace only an arc.

⚠ Use a cos t & b sin t, not a² & b²: a & b are semi-axis lengths, the square roots of the denominators.

✓ Check a parametrization: plug x(t) & y(t) into the implicit eqn; it must simplify to a true identity.

4.8 Vectors

Vector Basics

Vector: magnitude + direction; **v** = ⟨a, b⟩ = a**i** + b**j**. From P(x₁, y₁) to Q(x₂, y₂): ⟨x₂ − x₁, y₂ − y₁⟩ (end − start).

Magnitude |**v**| = √(a² + b²); **direction angle** θ: tan θ = b/a, then pick θ in the quadrant of (a, b). Ex.: ⟨3, 4⟩ → 5.

Components from length & angle: **v** = ⟨|**v**| cos θ, |**v**| sin θ⟩. Ex.: |**v**| = 10, θ = 30° → ⟨5√3, 5⟩ ≈ ⟨8.66, 5⟩.

Unit vector (length 1) in direction of **v**: **v**/|**v**|. Ex.: ⟨3, −4⟩ → ⟨3/5, −4/5⟩. **i** = ⟨1, 0⟩ & **j** = ⟨0, 1⟩ are unit vectors.

Equal vectors: same magnitude & direction, wherever they sit; position doesn't matter, only components do.

Operations

Operation	Rule	Picture
u + **v**	add components	head-to-tail
k**v**	⟨ka, kb⟩	scale by |k|
u − **v**	**u** + (−**v**)	tip of v → tip of u
u · **v**	$u_1v_1 + u_2v_2$	a scalar

k < 0 flips **v** to the opposite direction. **Resultant** of forces or velocities = their vector sum. Ex.: ⟨2, 1⟩ + ⟨−5, 3⟩ = ⟨−3, 4⟩.

Ex.: plane heads ⟨400, 0⟩ mph in a wind of ⟨0, 30⟩ → ground velocity ⟨400, 30⟩, speed √(400² + 30²) ≈ 401.1 mph.

Dot Product & Angles

Dot product: **u** · **v** = $u_1v_1 + u_2v_2$ = |**u**||**v**| cos θ, θ = angle b/w the vectors placed tail to tail (0 ≤ θ ≤ π).

Angle: cos θ = (**u** · **v**)/(|**u**||**v**|). Ex.: ⟨1, 0⟩ & ⟨1, 1⟩ → 1/(1·√2) = 1/√2 → θ = π/4; ⟨2, 1⟩ & ⟨−1, 2⟩ → θ = π/2.

Sign test: dot > 0 → acute; = 0 → **perpendicular** (orthogonal); < 0 → obtuse. Also **v** · **v** = |**v**|².

⚠ A dot product is a number, not a vector; and |u + v| ≠ |u| + |v| unless u & v point the same way.

✓ Sketch vectors tail to tail before finding an angle: it catches a wrong-quadrant θ from $\tan^{-1}$.

4.9 Vector-Valued Functions

Position Vectors

Vector-valued function: **p**(t) = ⟨x(t), y(t)⟩; tail at origin, tip traces the same path as the parametric (x(t), y(t)).

Distance from origin at time t = |**p**(t)| = √(x(t)² + y(t)²). Ex.: **p**(2) = ⟨3, 4⟩ → 5 units from the origin at t = 2.

Change & Velocity

Quantity	Formula
Position	**p**(t) = ⟨x(t), y(t)⟩
Displacement	**p**(b) − **p**(a)
Avg velocity	(**p**(b) − **p**(a))/(b − a)
Dist. from origin	|**p**(t)|

Displacement t₁ → t₂: **p**(t₂) − **p**(t₁), arrow from old to new position; its length = straight-line distance.

Avg rate of change (avg velocity) = ⟨Δx/Δt, Δy/Δt⟩; its direction = direction of travel b/w the 2 times.

Magnitude of avg velocity = displacement length ÷ Δt; it is ≤ avg speed b/c a curved path is longer than the chord.

Ex.: **p**(t) = ⟨t², 2t⟩ on [1, 3]: **p**(1) = ⟨1, 2⟩, **p**(3) = ⟨9, 6⟩ → displacement ⟨8, 4⟩, avg velocity ⟨4, 2⟩, magnitude 2√5.

Motion Analysis

Components act alone: x(t) ↑ → moving right; y(t) ↑ → moving up; one motion, but read x & y separately.

Projectile (ft, s): **p**(t) = ⟨(v_0 cos θ)t, (v_0 sin θ)t − 16t²⟩; horizontal speed constant, vertical slows, stops, reverses.

Circular motion: **p**(t) = ⟨r cos(ωt), r sin(ωt)⟩; ω = angular speed (rad per unit t), period 2π/ω, |**p**(t)| = r always.

Ex.: r = 4, ω = π/2 → one lap every 4 s; at t = 1, **p** = ⟨0, 4⟩, the top of the circle, a quarter-lap in from ⟨4, 0⟩.

⚠ Displacement ≠ distance traveled: one full lap of a circle has displacement ⟨0, 0⟩ but distance 2πr.

✓ Calc BC preview: replace these average rates w/ derivatives to get velocity & speed at an instant.

4.10–4.11 Matrices, Determinant & Inverse

Matrix Basics (rows split by ;)

Matrix: rectangular array; m × n = m rows × n columns; a_{ij} = entry in row i, column j. [1 2; 3 4] is 2 × 2.

Add/subtract: same size only, entry by entry. **Scalar** kA: multiply every entry by k. Ex.: 2[1 −3] = [2 −6].

Product AB: needs cols of A = rows of B; (m × n)(n × p) → m × p. Entry (i, j) = row i of A · column j of B.

Ex.: [1 2; 3 4][5 6; 7 8] = [19 22; 43 50]; e.g. 19 = 1·5 + 2·7 (row 1 · column 1) & 50 = 3·6 + 4·8 (row 2 · column 2).

Not commutative: AB ≠ BA in general, so keep the order.

Identity I = [1 0; 0 1]: AI = IA = A, like multiplying by 1.

Determinant

det [a b; c d] = ad − bc (main diagonal product minus the other diagonal's). Ex.: det [3 1; 4 2] = 3·2 − 1·4 = 2.

Area: |det| = area of the parallelogram formed by the matrix's 2 column vectors; det = 0 → the vectors are parallel.

Ex.: vectors ⟨3, 0⟩ & ⟨1, 2⟩ → det [3 1; 0 2] = 6 → parallelogram area 6 (base 3, height 2).

Inverse

Inverse: A^{-1} = (1/(ad − bc))[d −b; −c a]: swap a & d, negate b & c, divide by det. Then $AA^{-1} = A^{-1}A = I$.

Exists only if det ≠ 0 (A is **invertible**); det = 0 → **singular**, no inverse, and AX = B has no unique solution.

Ex.: A = [3 1; 4 2], det 2 → A^{-1} = [1 −1/2; −2 3/2]. Check: row 1 of A · column 1 of A^{-1} = 3 − 2 = 1 ✓.

Systems: AX = B → X = A^{-1}B (multiply on the left of both sides; BA^{-1} is a different matrix or undefined).

⚠ $(AB)^{-1} = B^{-1}A^{-1}$, not $A^{-1}B^{-1}$: reverse the order when undoing a product, like taking off socks then shoes.

✓ Check dimensions first: (2 × 3)(3 × 1) works → 2 × 1 result; (2 × 3)(2 × 3) is undefined b/c inner sizes 3 ≠ 2.

4.12–4.14 Linear Transformations & Models

Linear Transformations

Linear transformation L(x, y) = (ax + by, cx + dy) ↔ matrix [a b; c d]; L(**v**) = A**v**, w/ **v** written as a column.

Read the matrix: column 1 = image of ⟨1, 0⟩, column 2 = image of ⟨0, 1⟩. The origin always maps to itself.

Transformation	Matrix
Rotate θ CCW	[cos θ −sin θ; sin θ cos θ]
Reflect over x-axis	[1 0; 0 −1]
Reflect over y = x	[0 1; 1 0]
Dilate by k	[k 0; 0 k]

Matrices as Functions

Composition: apply B, then A → matrix AB (rightmost acts first). Ex.: rotate 90° twice = rotate 180° = −I.

Inverse map: A^{-1} undoes A. **|det A|** = area scale factor; det < 0 → orientation flips (includes a reflection).

det = 0: the plane collapses onto a line or a pt → info lost, so the map can't be undone (no inverse).

Modeling w/ Matrices

Transition matrix: entry = probability of moving from one state to another per step; each state's outflows sum to 1.

Predict: columns sum to 1 → $state_n = A^n(state_0)$, a column vector; rows sum to 1 → row vector × A^n.

Ex.: A = [0.9 0.2; 0.1 0.8], start ⟨100, 0⟩ → next step ⟨90, 10⟩, then ⟨83, 17⟩ (each column sums to 1).

Past states: multiply by A^{-1}. **Long run**: A^n**x** often settles at a **steady state x** where A**x** = **x** (no further change).

⚠ AB means do B first: rotate-then-reflect ≠ reflect-then-rotate, so write the matrices right to left.

✓ Steady state above: 0.1x = 0.2y & x + y = 100 → ⟨66.7, 33.3⟩; solve Ax = x w/ the total fixed.